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Equations of motion of test particles for solving the spin-dependent Boltzmann-Vlasov equation

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outline

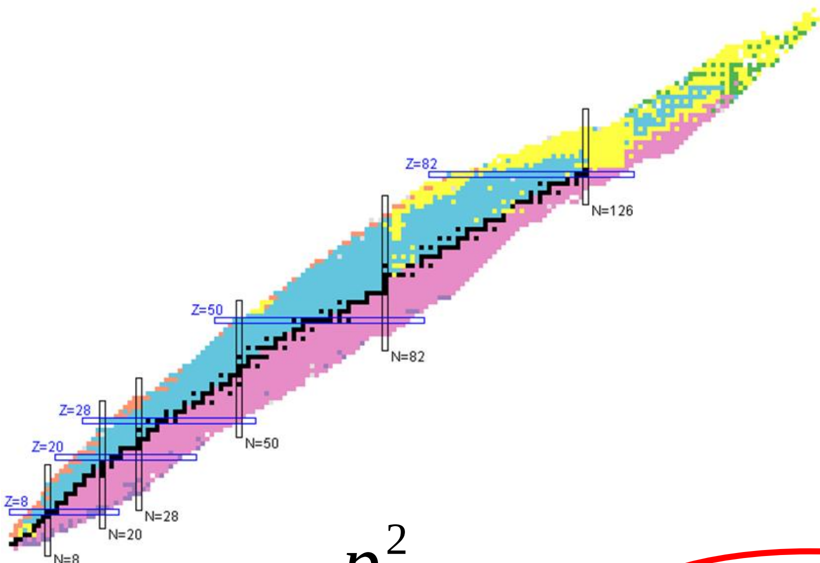
1. Background and motivation.

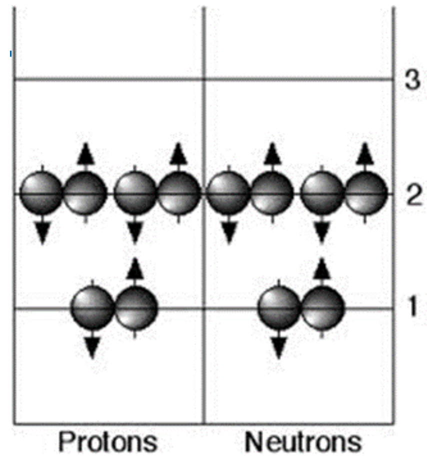
2. The derivation of equations of motion for the spin-dependent Boltzmann-Vlasov equation in HICs.

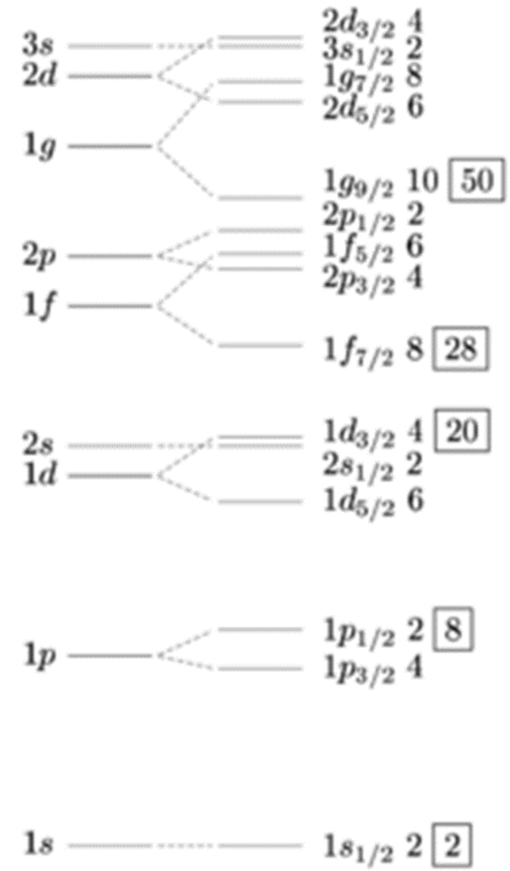
3. The spin splitting of the collective flows.

4. Conclusions & Outlook

The importance of nucleon spin degree of freedom







$$h_q = \frac{p^2}{2m} + U_q + \vec{W}_q \cdot (\vec{p} \times \vec{\sigma}), (q = n, p)$$

Schrödinger equation: $h_q \varphi_q = e_q \varphi_q$

The spin-orbit potential $\vec{W}_q \cdot (\vec{p} \times \vec{\sigma})$

help to explain the magic number and shell structure!

the role of nucleon spin is much less known in nuclear reactions than structures.

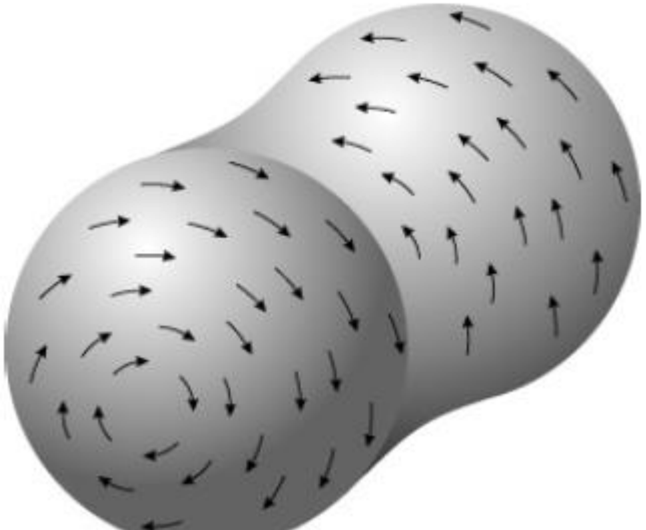
Spin-orbit potential at low and high energies

low energies (TDHF):

TABLE I. Thresholds for the inelastic scattering of $^{16}O + ^{16}O$ system.

Force	Skyrme II (MeV)	Skyrme M* (MeV)
Spin orbit	68	70
No spin orbit	31	27

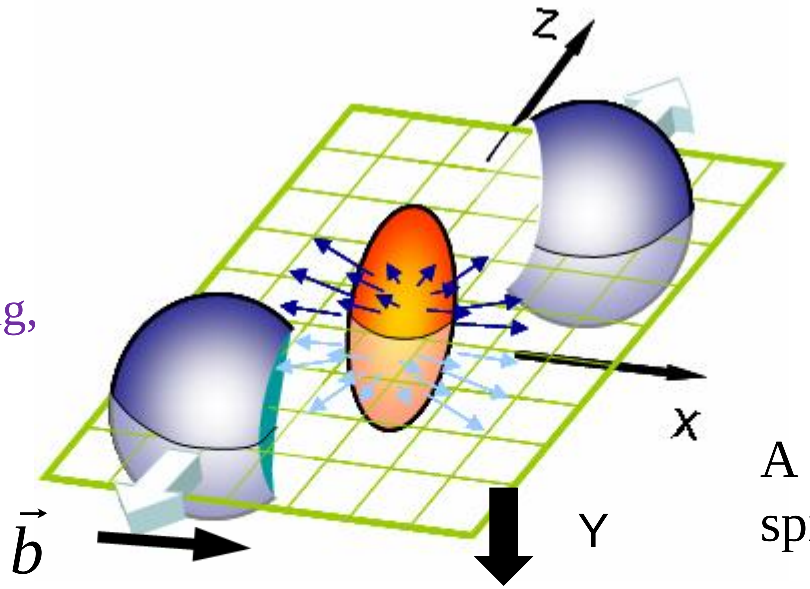
A. S. Umar *et al.*, Phys. Rev. Lett., 1986



J. A. Maruhn *et al*, Phys. Rev. C, 2006

high energies :

Z. T. Liang and X. N. Wang,
Phys. Rev. Lett., 2005
Phys. Lett. B, 2005



$$\vec{n}_{re} = \frac{\vec{p}_{in} \times \vec{b}}{\left|\vec{p}_{in} \times \vec{b}\right|}$$

A global quark
spin polarization

Motivation of this work

In solid state physics, two main methods of spin transport to study spin dynamics

I. Start from a model Hamiltonian

J. Sinova *et al.*, PRL 92, 126603 (2004)
G. Sundaram *et al.*, PRB 59 14915 (1999)

the canonical EOMs

$$\begin{cases} \frac{d\vec{r}}{dt} = \nabla_p \varepsilon \\ \frac{d\vec{p}}{dt} = -\nabla_r \varepsilon \end{cases}$$

II. Linearize the spin-dependent BV equation

through the relaxation time approximation.

K. Morawetz, PRB 92 245425 (2015)
J. W. Zhang *et al.*, PRL 93 256602 (2004)

commutation relation

$$\frac{d\vec{\sigma}}{dt} = \frac{1}{i}[\vec{\sigma}, \varepsilon]$$

In nuclear physics,

the test-particle method —***numerically solve*** (*the lowest order term*)→ spin-independent BV equation

C. Y. Wong, PRC 25, 1460 (1982)
G. F. Bertsch *et al*, PRC 29, 673 (1984).

the EOMs of test particles **are identical to** the canonical EOMs

What are EOMs for solving *spin-dependent* BV equation ?

The spin-dependent Boltzmann-Vlasov equation *(two reviews)*

Method I : from the BV equation with **a spinor distribution function**

$$\frac{\partial \hat{f}}{\partial t} + \frac{i}{\hbar}\left[\hat{\varepsilon}, \hat{f}\right] + \frac{1}{2}\left(\frac{\partial \hat{\varepsilon}}{\partial \vec{p}} \cdot \frac{\partial \hat{f}}{\partial \vec{r}} + \frac{\partial \hat{f}}{\partial \vec{r}} \cdot \frac{\partial \hat{\varepsilon}}{\partial \vec{p}}\right) - \frac{1}{2}\left(\frac{\partial \hat{\varepsilon}}{\partial \vec{r}} \cdot \frac{\partial \hat{f}}{\partial \vec{p}} + \frac{\partial \hat{f}}{\partial \vec{p}} \cdot \frac{\partial \hat{\varepsilon}}{\partial \vec{r}}\right) = 0$$

(the general BV equation: $\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \nabla_r f - \nabla_r U \cdot \nabla_p f = 0$ *.)*

H. Smith and H.H. Jensen, transport phenomena (Oxford University press, Oxford, 1989)

with $\hat{\varepsilon}$ and $\hat{f}$ can be expressed as

$$\hat{\varepsilon}(\vec{r},\vec{p}) = \varepsilon(\vec{r},\vec{p})\hat{I} + \vec{h}(\vec{r},\vec{p}) \cdot \vec{\sigma},$$
$$\hat{f}(\vec{r},\vec{p}) = f_0(\vec{r},\vec{p})\hat{I} + \vec{g}(\vec{r},\vec{p}) \cdot \vec{\sigma},$$

By substituting $\hat{\varepsilon}$ and $\hat{f}$ into spin-dependent BV equation :

scalar distribution

$$\frac{\partial f_0}{\partial t} + \frac{\partial \varepsilon}{\partial \vec{p}} \cdot \frac{\partial f_0}{\partial \vec{r}} - \frac{\partial \varepsilon}{\partial \vec{r}} \cdot \frac{\partial f_0}{\partial \vec{p}} + \frac{\partial \vec{h}}{\partial \vec{p}} \cdot \frac{\partial \vec{g}}{\partial \vec{r}} - \frac{\partial \vec{h}}{\partial \vec{r}} \cdot \frac{\partial \vec{g}}{\partial \vec{p}} = 0,$$

vector distribution

$$\frac{\partial \vec{g}}{\partial t} + \frac{\partial \varepsilon}{\partial \vec{p}} \cdot \frac{\partial \vec{g}}{\partial \vec{r}} - \frac{\partial \varepsilon}{\partial \vec{r}} \cdot \frac{\partial \vec{g}}{\partial \vec{p}} + \frac{\partial f_0}{\partial \vec{r}} \cdot \frac{\partial \vec{h}}{\partial \vec{p}} - \frac{\partial f_0}{\partial \vec{p}} \cdot \frac{\partial \vec{h}}{\partial \vec{r}} + \frac{2\vec{g} \times \vec{h}}{\hbar} = 0.$$

Method II : from Wigner transformation of TDHF equation with spin. ($i\hbar\dot{\hat{\rho}} = [\hat{h}, \hat{\rho}]$)

$$i\hbar\langle\vec{r},s|\dot{\hat{\rho}}|\vec{r}'',s''\rangle = \sum_{s'}\int d^3r'[\langle\vec{r},s|\hat{h}|\vec{r}',s'\rangle\langle\vec{r}',s'|\hat{\rho}|\vec{r}'',s''\rangle - \langle\vec{r},s|\hat{\rho}|\vec{r}',s'\rangle\langle\vec{r}',s'|\hat{h}|\vec{r}'',s''\rangle]$$

E.B. Balbutsev *et al*, NPA(2011); PRC(2013)

$s, s'', s' \to \uparrow or \downarrow$

Wigner transformation

$$f_{\uparrow\uparrow},\ f_{\uparrow\downarrow},\ f_{\downarrow\uparrow},\ f_{\downarrow\downarrow}$$

$$\begin{pmatrix} f_{\uparrow\uparrow} & f_{\downarrow\uparrow} \\ f_{\uparrow\downarrow} & f_{\downarrow\downarrow} \end{pmatrix}$$

The definition of the Wigner function of particles with spin-1/2:

$$f_{\sigma\sigma'}(\vec{r}\vec{p},t) = \int d^3s e^{i\vec{p}\cdot\vec{s}/\hbar}\psi_\sigma(\vec{r}-\frac{\vec{s}}{2},t)\psi_{\sigma'}(\vec{r}+\frac{\vec{s}}{2},t)$$

$$f(\vec{r}\vec{p},t,0) = f_{\uparrow\uparrow}(\vec{r}\vec{p},t) + f_{\downarrow\downarrow}(\vec{r}\vec{p},t)$$

scalar distribution

$$\tau(\vec{r}\vec{p},t,x) = f_{\downarrow\uparrow}(\vec{r}\vec{p},t) + f_{\uparrow\downarrow}(\vec{r}\vec{p},t)$$

$$\tau(\vec{r}\vec{p},t,y) = -i[f_{\downarrow\uparrow}(\vec{r}\vec{p},t) - f_{\uparrow\downarrow}(\vec{r}\vec{p},t)]$$

$$\tau(\vec{r}\vec{p},t,z) = f_{\uparrow\uparrow}(\vec{r}\vec{p},t) - f_{\downarrow\downarrow}(\vec{r}\vec{p},t)$$

Three components of vector distribution

R. F. O'Connell *et al*, PRA (1984)

Two methods give the same BV equation !

$$f(\vec{r}\vec{p},t,0) = 2f_0$$

$$\tau(\vec{r}\vec{p},t) = 2\vec{g}$$

Single-particle energy with spin-orbit interaction

Skyrme spin-orbit interaction:

$$V_{so} = i W_0(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \vec{k} \times \delta(\vec{r}_1 - \vec{r}_2)\vec{k}',$$

The spin-dependent single-particle energy: Y.M. Engel *et al.*, NPA (1975)

$$h_q^{so}(\vec{r}, \vec{p}) = h_1 + h_4 + (\vec{h}_2 + \vec{h}_3) \cdot \vec{\sigma}$$

$$\begin{aligned} h_1 &= -\frac{W_0}{2}\nabla_{\vec{r}} \cdot [\vec{J}(\vec{r}) + \vec{J}_q(\vec{r})], \\ \vec{h}_2 &= -\frac{W_0}{2}\nabla_{\vec{r}} \times [\vec{j}(\vec{r}) + \vec{j}_q(\vec{r})], \\ \vec{h}_3 &= \frac{W_0}{2}\nabla_{\vec{r}}[\rho(\vec{r}) + \rho_q(\vec{r})] \times \vec{p}, \\ h_4 &= -\frac{W_0}{2}\nabla_{\vec{r}} \times [\vec{s}(\vec{r}) + \vec{s}_q(\vec{r})] \cdot \vec{p}. \end{aligned}$$

$$\begin{aligned} \rho(\vec{r}) &= \int d^3p f(\vec{r}, \vec{p}), \\ \vec{s}(\vec{r}) &= \int d^3p \vec{\tau}(\vec{r}, \vec{p}), \\ \vec{j}(\vec{r}) &= \int d^3p \frac{\vec{p}}{\hbar} f(\vec{r}, \vec{p}), \\ \vec{J}(\vec{r}) &= \int d^3p \frac{\vec{p}}{\hbar} \times \vec{\tau}(\vec{r}, \vec{p}). \end{aligned}$$

Single-particle energy can be written as :

$$\begin{aligned} \varepsilon_q(\vec{r}, \vec{p}) &= \frac{p^2}{2m} + U_q + h_1 + h_4, \\ \vec{h}_q(\vec{r}, \vec{p}) &= \vec{h}_2 + \vec{h}_3, \end{aligned}$$

(U_q is the spin-independent mean-field potential.)

Spin-dependent EOMs of test particles

The vector part of the spinor Wigner function distribution:

$$\vec{g}(\vec{r},\vec{p}) = \vec{n} f_1(\vec{r},\vec{p}).$$

A unit vector $\vec{n}$

Substituting back into the spin-dependent BV equation and after some algebra,

$$\frac{\partial f_0}{\partial t} + \frac{\partial \varepsilon}{\partial \vec{p}} \cdot \frac{\partial f_0}{\partial \vec{r}} - \frac{\partial \varepsilon}{\partial \vec{r}} \cdot \frac{\partial f_0}{\partial \vec{p}} + \left(\frac{\partial \vec{h}}{\partial \vec{p}} \cdot \vec{n}\right) \cdot \frac{\partial f_1}{\partial \vec{r}} - \left(\frac{\partial \vec{h}}{\partial \vec{r}} \cdot \vec{n}\right) \cdot \frac{\partial f_1}{\partial \vec{p}} = 0$$

$$\frac{\partial f_1}{\partial t} + \frac{\partial \varepsilon}{\partial \vec{p}} \cdot \frac{\partial f_1}{\partial \vec{r}} - \frac{\partial \varepsilon}{\partial \vec{r}} \cdot \frac{\partial f_1}{\partial \vec{p}} + \frac{\partial f_0}{\partial \vec{r}} \cdot \left(\frac{\partial \vec{h}}{\partial \vec{p}} \cdot \vec{n}\right) - \frac{\partial f_0}{\partial \vec{p}} \cdot \left(\frac{\partial \vec{h}}{\partial \vec{r}} \cdot \vec{n}\right) = 0.$$

$$\frac{\partial \vec{n}}{\partial t} \approx \frac{2\vec{h} \times \vec{n}}{\hbar}.$$

These two equations can be **decoupled**, $V_{hn} = \vec{h} \cdot \vec{n}$

$$\frac{\partial (f^+)}{\partial t} + \left(\frac{\partial \varepsilon + \partial V_{hn}}{\partial \vec{p}}\right) \cdot \frac{\partial (f^+)}{\partial \vec{r}} - \left(\frac{\partial \varepsilon + \partial V_{hn}}{\partial \vec{r}}\right) \cdot \frac{\partial (f^+)}{\partial \vec{p}} = 0$$

$$\frac{\partial (f^-)}{\partial t} + \left(\frac{\partial \varepsilon - \partial V_{hn}}{\partial \vec{p}}\right) \cdot \frac{\partial (f^-)}{\partial \vec{r}} - \left(\frac{\partial \varepsilon - \partial V_{hn}}{\partial \vec{r}}\right) \cdot \frac{\partial (f^-)}{\partial \vec{p}} = 0$$

with $f^{\pm} = f_0 \pm f_1$

arXiv:1602.00404

The equation for $f^{\pm}$ is just **a standard Vlasov equation** with the single-particle Hamiltonian $\varepsilon \pm V_{hn}$.

Obviously $f^{\pm}$ represent the phase-space distributions of the particles with their spin in $\pm\vec{n}$ directions.

$$\hat{f}(\vec{r},\vec{p}) = f_0(\vec{r},\vec{p})\hat{I} + f_1(\vec{r},\vec{p})\vec{n}\cdot\vec{\sigma} \xrightarrow{\text{eigenfunction}} \begin{cases} f^{+} = f_0 + f_1 & \text{spin-up} \\ f^{-} = f_0 - f_1 & \text{spin-down} \end{cases} \quad +\vec{n}$$

Introduce **two type test-particles** **to *independently* solve each of these equations**

arbitrary function

$$f^{\pm}(\vec{r},\vec{p},t) = \int \frac{d^3r_0 d^3p_0 d^3s}{(2\pi\hbar)^3} \exp\{i\vec{s}\cdot[\vec{p}-\vec{P}(\vec{r}_0\vec{p}_0\vec{s},t)]/\hbar\} \times \delta[\vec{r}-\vec{R}(\vec{r}_0\vec{p}_0\vec{s},t)] f^{\pm}(\vec{r}_0,\vec{p}_0,t_0)$$

with the initial conditions $\vec{R}(\vec{r}_0\vec{p}_0\vec{s},t_0) = \vec{r}_0$ and $\vec{P}(\vec{r}_0\vec{p}_0\vec{s},t_0) = \vec{p}_0$

find the new phase-space coordinates $\vec{R}(\vec{r}_0\vec{p}_0\vec{s},t)$ and $\vec{P}(\vec{r}_0\vec{p}_0\vec{s},t)$ at $t = t_0 + \Delta t$

Substitute into the decoupled equation

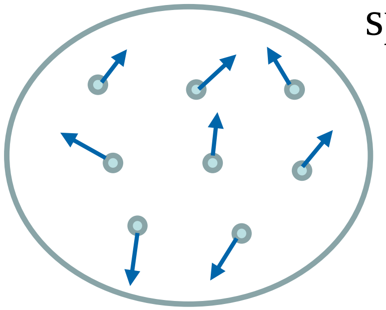
$$\begin{cases} \dfrac{\partial\vec{R}}{\partial t} = \dfrac{\partial\varepsilon}{\partial\vec{p}} \pm \dfrac{\partial V_{hn}}{\partial\vec{p}} \\ \dfrac{\partial\vec{P}}{\partial t} = -\dfrac{\partial\varepsilon}{\partial\vec{r}} \mp \dfrac{\partial V_{hn}}{\partial\vec{r}} \end{cases}$$

follow the same method as in C. Y. Wong's paper

C. Y. Wong, PRC 25, 1460 (1982)

→ **two EOMs for two distributions of the particles $f^{\pm}$,respectively**

Spin direction is arbitrary and do not set a spin reference direction



spin vector $\langle S\rangle=[\langle S_x\rangle,\langle S_y\rangle,\langle S_z\rangle]$

$\vec{n}\cdot\vec{\sigma}$ —spin along $\vec{n}$ direction→ $\langle\vec{\sigma}\rangle=\vec{n}$

f^+ and f^- represent **the same type** of phase-space distributions

EOMs

$$\frac{\partial\vec{R}}{\partial t}=\frac{\partial\varepsilon}{\partial\vec{p}}+\frac{\partial V_{hn}}{\partial\vec{p}}$$

$$\frac{\partial\vec{P}}{\partial t}=-\frac{\partial\varepsilon}{\partial\vec{r}}-\frac{\partial V_{hn}}{\partial\vec{r}}$$

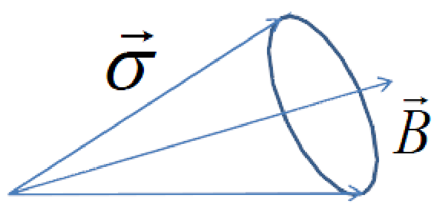
$$\frac{\partial\vec{n}}{\partial t}=\frac{2\vec{h}\times\vec{n}}{\hbar}$$

similar to the *canonical equations*

$$\frac{d\vec{r}}{dt}=\nabla_p\varepsilon$$

$$\frac{d\vec{p}}{dt}=-\nabla_r\varepsilon$$

$$\frac{d\vec{\sigma}}{dt}=\frac{1}{i}[\vec{\sigma},\varepsilon]$$



$$\varepsilon\sim-\vec{\sigma}\cdot\vec{B}$$

$$\frac{d\vec{\sigma}}{dt}\sim\vec{\sigma}\times\vec{B}$$

This kind of treatment is the same as our previous work.

J. Xu *et al*, Phys. Lett. B(2013)
Y. Xia *et al*, Phys. Rev. C(2014)
J. Xu *et al*, Frontiers of Physics (2015)

Single-particle energy $\varepsilon + \vec{h}\cdot\vec{n}$ — *Similar to* → $\vec{\sigma}\cdot\vec{B}$ external magnetic field

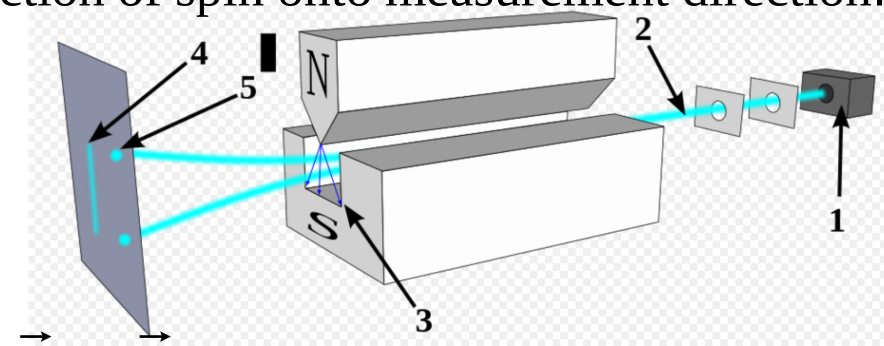
(a specific reference direction ?)

do not commute with each other.

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k \,(ijk \sim xyz)$$

→ Three components of spin states can not be measured **simultaneously**.

Similar to the Stern-Gerlach experiment, projection of spin onto measurement direction.

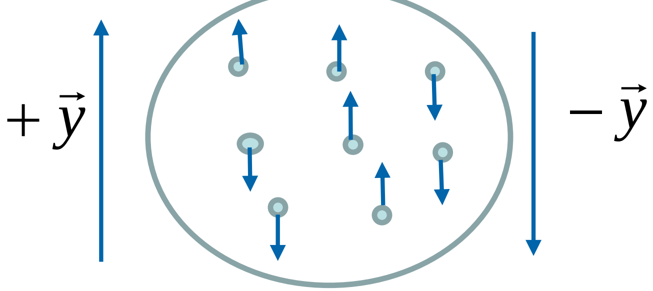


$$h_y >> h_x or h_z$$

(a limit case)

(omit spin evolution)

set $\vec{y}$ as a specific reference direction , $\vec{n} = \vec{y}$



$\vec{n}_i = +\vec{y}$ for spin-up

$\vec{n}_i = -\vec{y}$ for spin-down

(similar to isospin case, proton and neutron distribution in same system)

Different EOMs for particles with different spin state (in the third spin direction)

EOMs

$$\begin{cases} \dfrac{\partial \vec{R}}{\partial t} = \dfrac{\partial \varepsilon}{\partial \vec{p}} + \dfrac{\partial h_y}{\partial \vec{p}} \\ \dfrac{\partial \vec{P}}{\partial t} = -\dfrac{\partial \varepsilon}{\partial \vec{r}} - \dfrac{\partial h_y}{\partial \vec{r}} \end{cases}$$

spin-up particles with the single-particle Hamiltonian $\varepsilon + h_y$

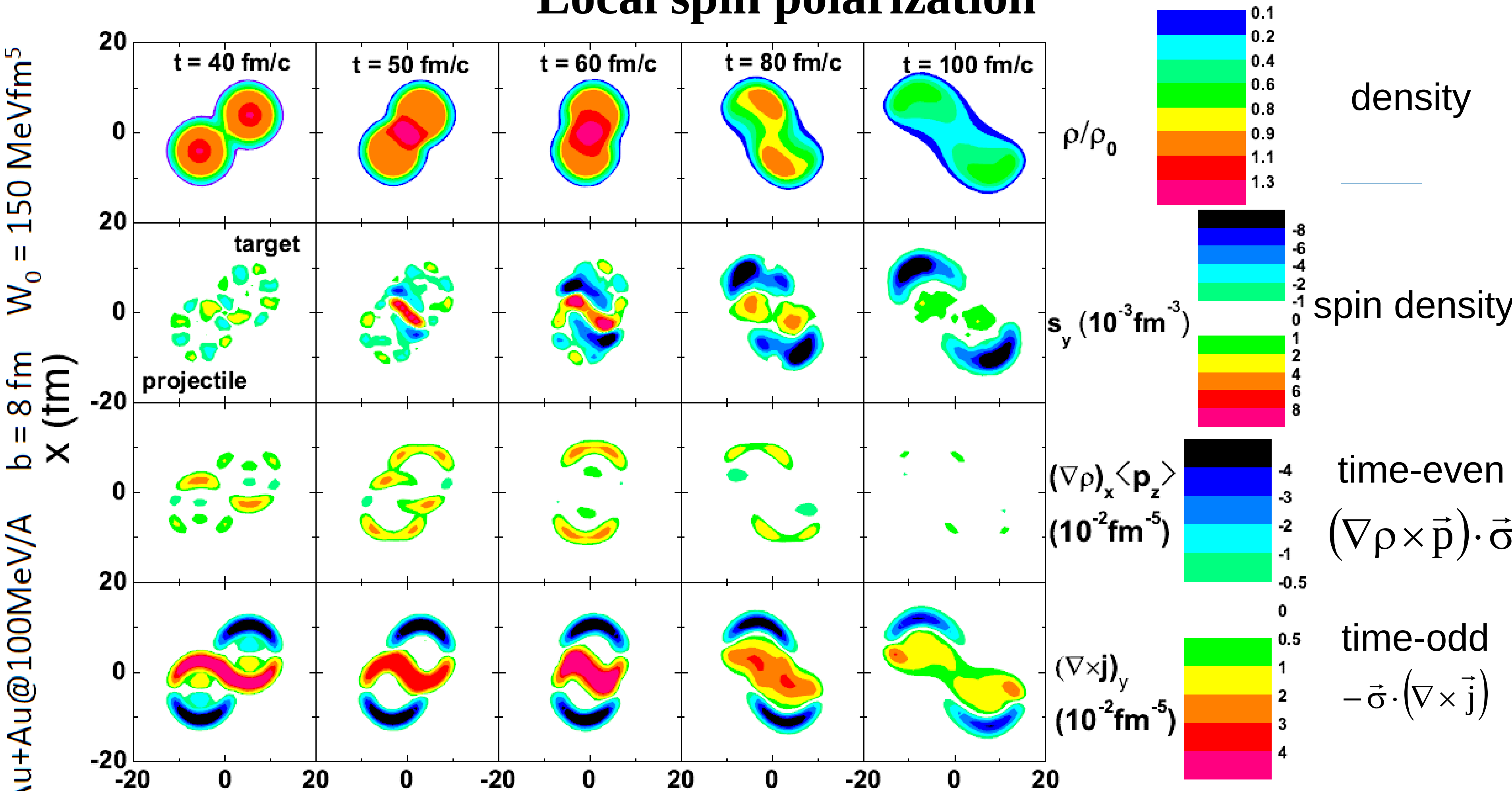
$$\begin{cases} \dfrac{\partial \vec{R}}{\partial t} = \dfrac{\partial \varepsilon}{\partial \vec{p}} - \dfrac{\partial h_y}{\partial \vec{p}} \\ \dfrac{\partial \vec{P}}{\partial t} = -\dfrac{\partial \varepsilon}{\partial \vec{r}} + \dfrac{\partial h_y}{\partial \vec{r}} \end{cases}$$

spin-down particles with the single-particle Hamiltonian $\varepsilon - h_y$

The force acting on a test particle *depends on* spin up or spin down.

It may be more suitable to describes properly *the correlation between the spin and the trajectory in measurement*! *(but it omits spin evolution)*

Local spin polarization



$(\nabla \times j)_y >> (\nabla \times j)_x or (\nabla \times j)_z$

arXiv:1411.3057

participant : +y
spectator : -y

A local spin polarization is observed.
Time-odd terms overwhelm time-even terms.

The spin splitting of transverse flow

Transverse flow $\langle p_x \rangle \sim y$

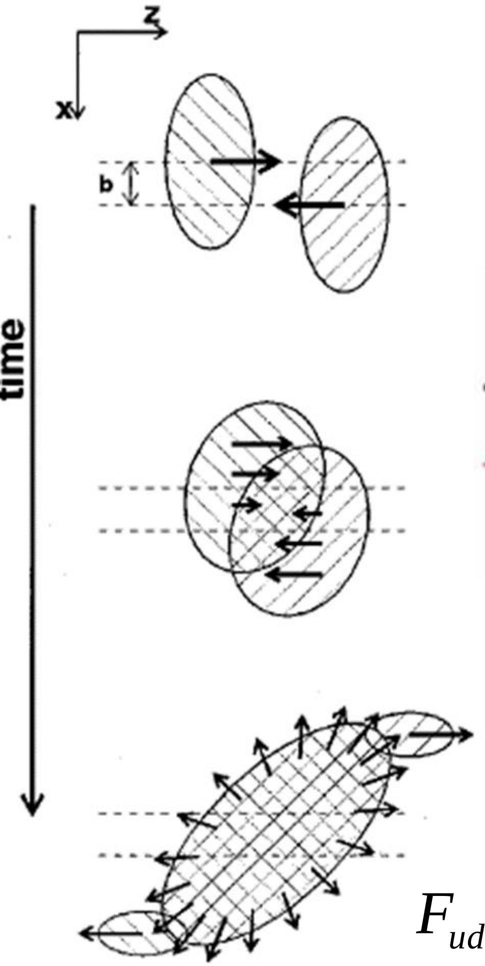
sensitive to nuclear interaction

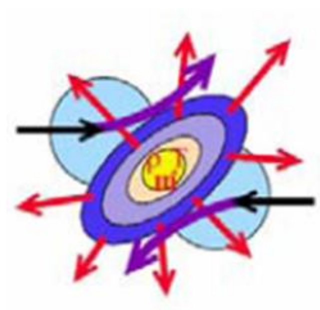
$$U = U_0 + \sigma U_{spin}$$

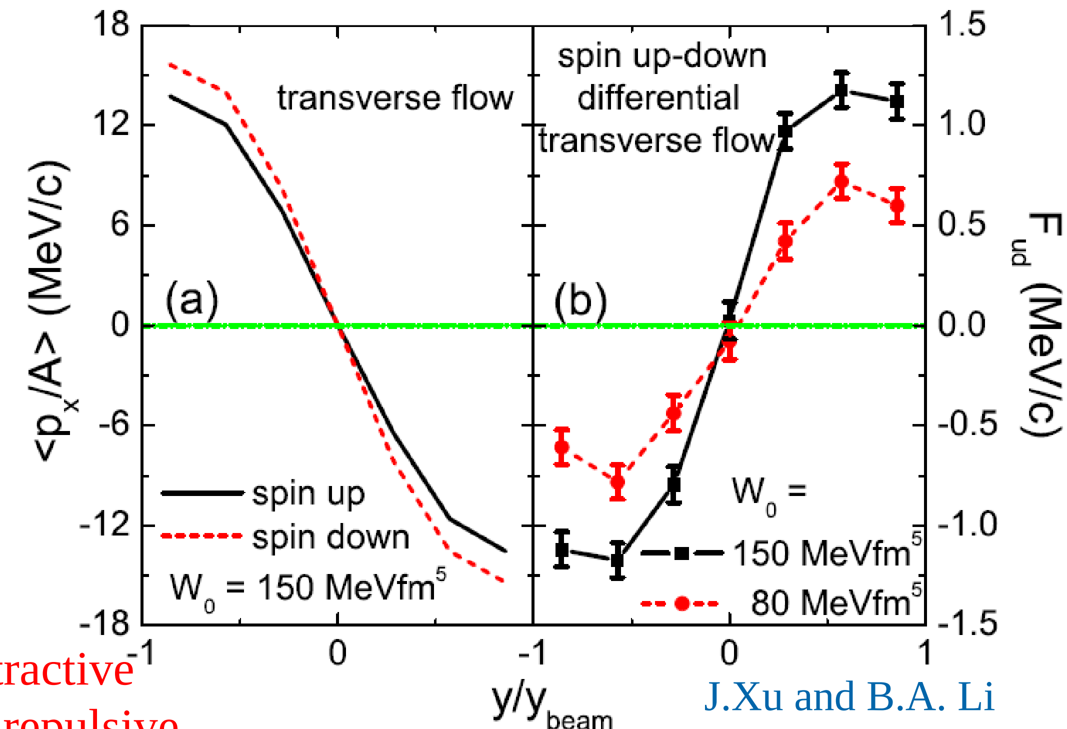
$$\sigma = 1(\uparrow) or -1(\downarrow)$$

$$F_{ud}(y) = \frac{1}{N(y)} \sum_{i=1}^{N(y)} \sigma_i (p_x)_i$$

reflects different transverse flows of spin-up and spin-down nucleons







spin-up (+y): attractive

spin-down (-y): repulsive

J.Xu and B.A. Li PLB (2013)

Transverse flow **larger** for spin down nucleons

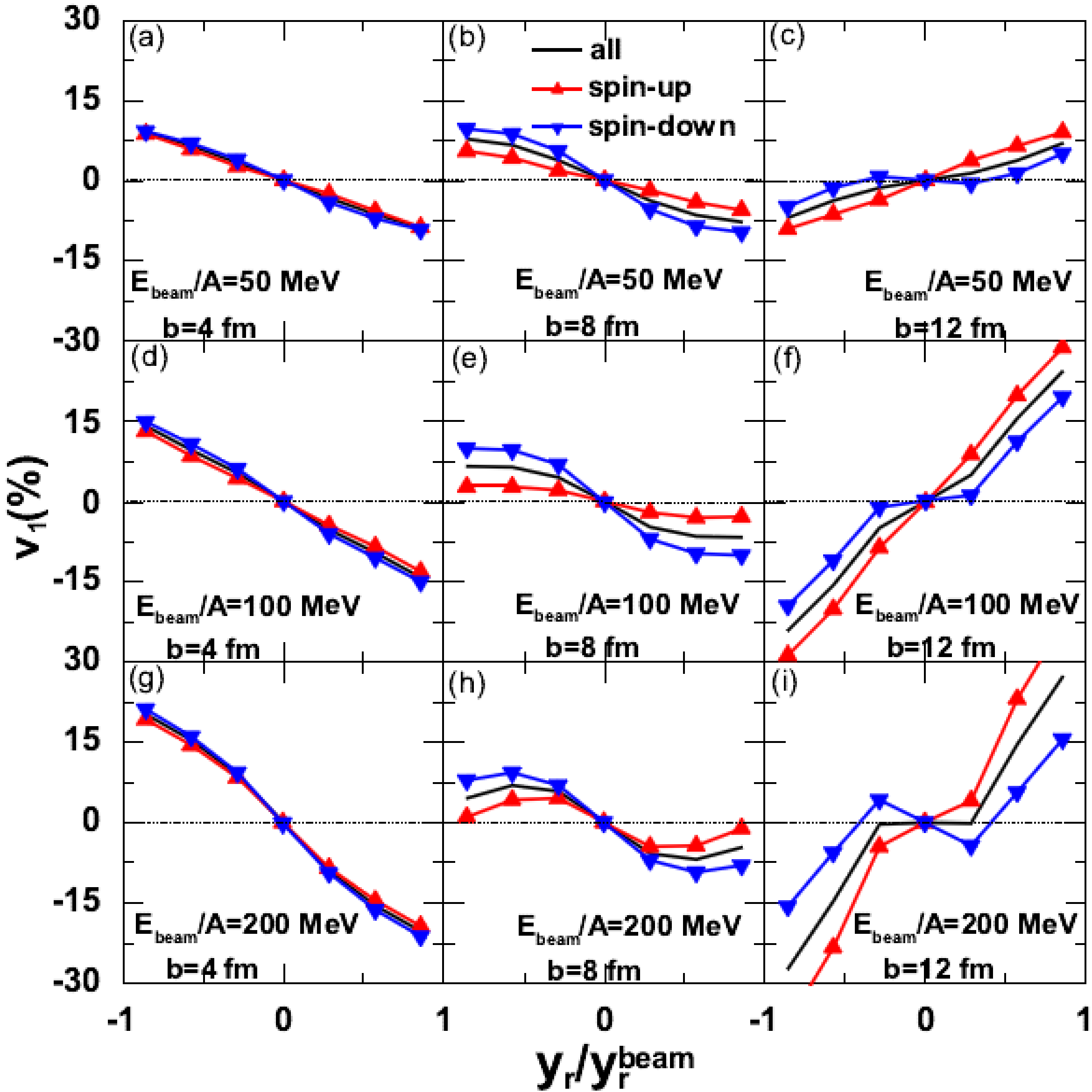
F_{ud} is sensitive to the strength of the spin-orbit interaction

The spin splitting of directed flow v_1

Directed flow:

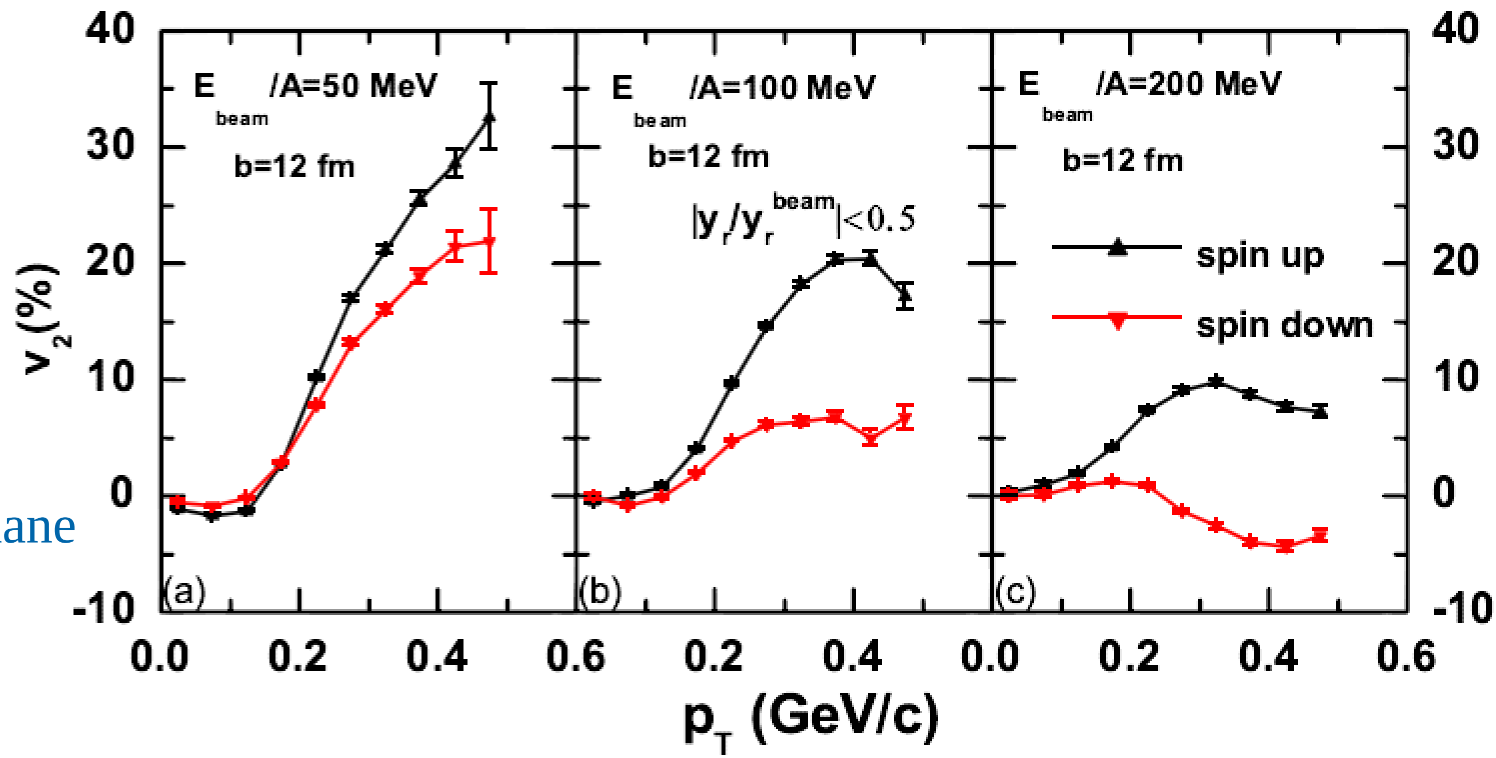
$$v_1 = \langle \cos(\phi) \rangle = \left\langle \frac{p_x}{p_T} \right\rangle$$

Spin splitting is **obvious** at different beam energy and impact parameter.



Effects of spin-orbit interaction on elliptic flow v_2

+: in-plane
-: out-of-plane



Elliptic flow:

$$v_2 = \langle \cos(2\phi) \rangle = \left\langle \frac{p_x^2 - p_y^2}{p_x^2 + p_y^2} \right\rangle$$

The spin splitting of elliptic flow increases with increasing nucleon momentum

The multiplicity of a M-nucleon cluster is

$$N_M = G \int \sum_{i_1 > i_2 > \ldots > i_M} d\mathbf{r}_{i_1} d\mathbf{k}_{i_1} \cdots d\mathbf{r}_{i_{M-1}} d\mathbf{k}_{i_{M-1}} \times \langle \rho_i^W (\mathbf{r}_{i_1}, \mathbf{k}_{i_1} \cdots \mathbf{r}_{i_{M-1}}, \mathbf{k}_{i_{M-1}}) \rangle .$$

R. Mattiello et al.,
Phys. Rev. Lett 1995
Phys. Rev. C 1997.

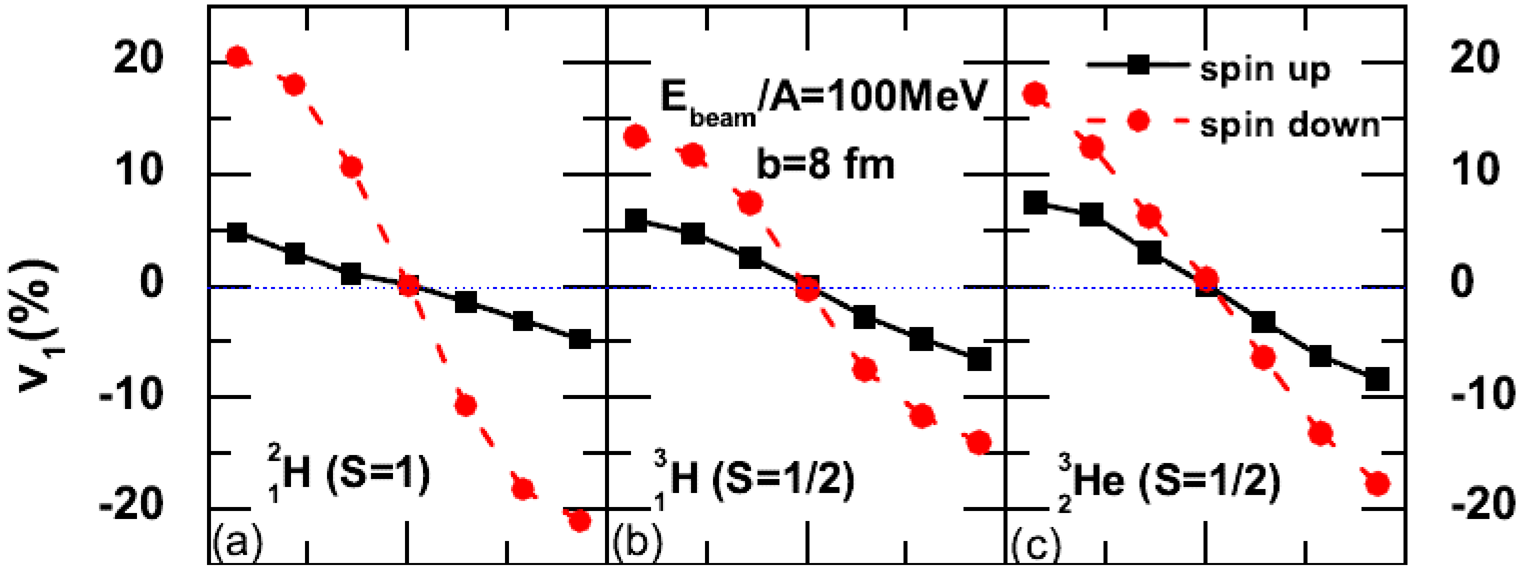
ρ^W is the Wigner phase-space density of the M-nucleon cluster,

Angular momentum conservation? Spatial wave function: s-wave assumption

- G : coalescence with **a given isospin**
- G': coalescence with **a given spin and isospin**

G: 3/8, d; 1/12, t; 1/12, ^3He

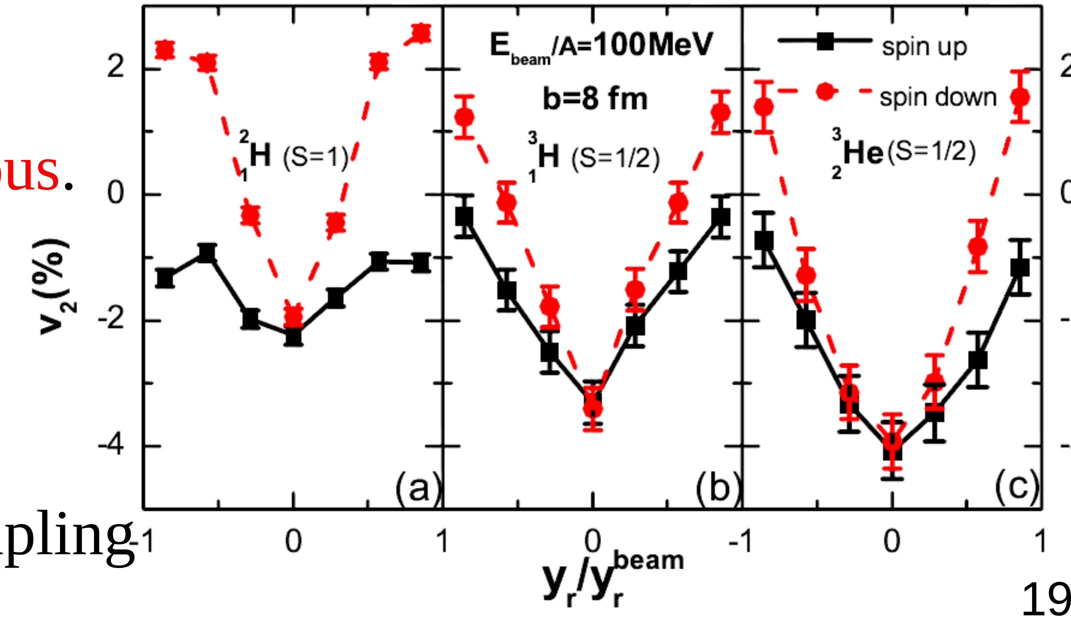
${}^2_1H(S=1)$ G'	${}^3_1H(S=1/2)$ G'	${}^3_2He(S=1/2)$ G'
$p\uparrow \& n\uparrow \rightarrow 1/2\ (S_z=1)$	$p\uparrow \& n\uparrow \& n\downarrow \rightarrow 1/3\ (S_z=1/2)$	$n\uparrow \& p\uparrow \& p\downarrow \rightarrow 1/3\ (S_z=1/2)$
$p\uparrow \& n\downarrow \rightarrow 1/4\ (S_z=0)$	$p\downarrow \& n\uparrow \& n\downarrow \rightarrow 1/3\ (S_z=-1/2)$	$n\downarrow \& p\uparrow \& p\downarrow \rightarrow 1/3\ (S_z=-1/2)$
$p\downarrow \& n\uparrow \rightarrow 1/4\ (S_z=0)$		
$p\downarrow \& n\downarrow \rightarrow 1/2\ (S_z=-1)$		



Spin-splitting of light clusters collective flows is more obvious.

Easily experimentally measured/identified

Useful probe of spin-orbit coupling



Conclusions & Outlook

I. We derive equations of motion (EOMs) of nucleon test particles for solving the spin-dependent BUU equation for the first time.

II. Considering further the quantum nature of spin, the EOMs of spin-up and spin-down nucleons are given separately in reference direction. (without spin evolution)

III. We study the spin splitting of the collective flows. It may be a sensitive probe of SO coupling in HICs.

IV. Hope future comparisons of model simulations with experimental data will help constrain the poorly known in-medium nucleon spin-orbit coupling.

Thank you for attention!

Back up

Test-particles method

$$f^{\pm}(\vec{r},\vec{p},t) = \int \frac{d^3 r_0 d^3 p_0 d^3 s}{(2\pi\hbar)^3} \exp\{i\vec{s}\cdot[\vec{p}-\vec{P}(\vec{r}_0\vec{p}_0\vec{s},t)]/\hbar\} \times \delta[\vec{r}-\vec{R}(\vec{r}_0\vec{p}_0\vec{s},t)] f^{\pm}(\vec{r}_0,\vec{p}_0,t_0), \quad (33)$$

Substituting back into the decoupled Vlasov equation,

$$\begin{aligned}
&[- \frac{\partial \vec{R}(\vec{r}_0\vec{p}_0\vec{s},t)}{\partial t} + \frac{\partial \varepsilon}{\partial \vec{p}}] \cdot \frac{\partial f^{\pm}(\vec{r},\vec{p},t)}{\partial \vec{r}} \pm \frac{\partial V_{hn}}{\partial \vec{p}} \cdot \frac{\partial f^{\pm}(\vec{r},\vec{p},t)}{\partial \vec{r}} \\
&+ \int \frac{d^3 r_0 d^3 p_0 d^3 s}{(2\pi\hbar)^3} \{ f^{\pm}(\vec{r}_0,\vec{p}_0,t_0)[\frac{-i\vec{s}}{\hbar} \cdot \frac{\partial \vec{P}(\vec{r}_0\vec{p}_0\vec{s},t)}{\partial t} \\
&- \frac{[\varepsilon(\vec{r}-\frac{\vec{s}}{2},t) - \varepsilon(\vec{r}+\frac{\vec{s}}{2},t)]}{i\hbar}] \\
&\mp f^{\pm}(\vec{r}_0,\vec{p}_0,t_0)[\frac{[V_{hn}(\vec{r}-\frac{\vec{s}}{2},t) - V_{hn}(\vec{r}+\frac{\vec{s}}{2},t)]}{i\hbar}]\} \\
&\times \exp\{i\vec{s}\cdot[\vec{p}-\vec{P}(\vec{r}_0\vec{p}_0\vec{s},t)]/\hbar\} \\
&\times \delta[\vec{r}-\vec{R}(\vec{r}_0\vec{p}_0\vec{s},t)] = 0, \quad (34)
\end{aligned}$$

EOMs from test-particle method

$$\left[-\frac{\partial \vec{R}(\vec{r}_0\vec{p}_0\vec{s},t)}{\partial t}+\frac{\partial \varepsilon}{\partial \vec{p}}\right]\cdot\frac{\partial f^{\pm}(\vec{r},\vec{p},t)}{\partial \vec{r}} \pm \frac{\partial V_{hn}}{\partial \vec{p}}\cdot\frac{\partial f^{\pm}(\vec{r},\vec{p},t)}{\partial \vec{r}} = 0, \qquad (35)$$

$$\frac{\partial \vec{R}(\vec{r}_0\vec{p}_0\vec{s},t)}{\partial t}=\frac{\partial \varepsilon}{\partial \vec{p}}\pm\frac{\partial V_{hn}}{\partial \vec{p}}$$

$$f^{\pm}(\vec{r}_0,\vec{p}_0,t_0)\{\frac{-i\vec{s}}{\hbar}\cdot\frac{\partial \vec{P}(\vec{r}_0\vec{p}_0\vec{s},t)}{\partial t} - \frac{[\varepsilon(\vec{r}-\frac{\vec{s}}{2},t)-\varepsilon(\vec{r}+\frac{\vec{s}}{2},t)]}{i\hbar}\}\mp f^{\pm}(\vec{r}_0,\vec{p}_0,t_0) \times\{\frac{[V_{hn}(\vec{r}-\frac{\vec{s}}{2},t)-V_{hn}(\vec{r}+\frac{\vec{s}}{2},t)]}{i\hbar}\}=0. \qquad (36)$$

$$\vec{s}\cdot\frac{\partial \vec{P}(\vec{r}_0\vec{p}_0\vec{s},t)}{\partial t}=[\varepsilon(\vec{r}-\frac{\vec{s}}{2},t)-\varepsilon(\vec{r}+\frac{\vec{s}}{2},t) \pm[V_{hn}(\vec{r}-\frac{\vec{s}}{2},t)-V_{hn}(\vec{r}+\frac{\vec{s}}{2},t)$$

Keep lowest order

$$\frac{\partial \vec{P}(\vec{r}_0\vec{p}_0\vec{s},t)}{\partial t}=-\frac{\partial \varepsilon}{\partial \vec{r}}\mp\frac{\partial V_{hn}}{\partial \vec{r}}.$$

Wigner phase-space density

deuteron

$$\rho_d^W(\mathbf{r},\mathbf{k}) = \int \phi\left(\mathbf{r}+\frac{\mathbf{R}}{2}\right)\phi^*\left(\mathbf{r}-\frac{\mathbf{R}}{2}\right)\exp(-i\mathbf{k}\cdot\mathbf{R})\,d\mathbf{R},$$

$$\mathbf{k} = (\mathbf{k}_1 - \mathbf{k}_2)/2 \qquad \mathbf{r} = (\mathbf{r}_1 - \mathbf{r}_2)$$

Internal wave function $\phi(r)$ ➡ root-mean-square radius of 1.96 fm

Triton or Helium3

$$\rho^W_{\mathrm{t}(^3\mathrm{He})}(\rho,\lambda,\mathbf{k}_\rho,\mathbf{k}_\lambda) = \int \psi\left(\rho+\frac{\mathbf{R}_1}{2},\lambda+\frac{\mathbf{R}_2}{2}\right)\psi^*\left(\rho-\frac{\mathbf{R}_1}{2},\lambda-\frac{\mathbf{R}_2}{2}\right)$$
$$\times\exp(-i\mathbf{k}_\rho\cdot\mathbf{R}_1)\exp(-i\mathbf{k}_\lambda\cdot\mathbf{R}_2)3^{3/2}\,d\mathbf{R}_1\,d\mathbf{R}_2$$

$$\begin{pmatrix}\mathbf{R}\\ \boldsymbol{\rho}\\ \boldsymbol{\lambda}\end{pmatrix} = J\begin{pmatrix}\mathbf{r}_1\\ \mathbf{r}_2\\ \mathbf{r}_3\end{pmatrix} \quad J = \begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\end{pmatrix} \quad \begin{pmatrix}\mathbf{K}\\ \mathbf{k}_\rho\\ \mathbf{k}_\lambda\end{pmatrix} = J^{-,+}\begin{pmatrix}\mathbf{k}_1\\ \mathbf{k}_2\\ \mathbf{k}_3\end{pmatrix} \quad J^{-,+} = \begin{pmatrix}1 & 1 & 1\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\end{pmatrix}$$

Internal wave function $\psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ ➡ RMS radius 1.61 and 1.74 fm for triton and ^{3}He.

S=1 T=0

${}^{2}_{1}H$ wave function

$$\left|{}^{2}_{1}H\right\rangle \sim \left|spin\right\rangle\left|isospin\right\rangle$$

S		T	
$S=1$	$\uparrow\uparrow$	pp	$T=1$
	$\frac{1}{\sqrt{2}}(\uparrow\downarrow+\downarrow\uparrow)$	$\frac{1}{\sqrt{2}}(pn+np)$	
	$\downarrow\downarrow$	nn	
$S=0$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)$	$\frac{1}{\sqrt{2}}(pn-np)$	$T=0$

Assign all many-nucleon states which are allowed from the Pauli principle the same weight .

8 wave function(considering the spin-isospin and exchange of antisymmetric), 3 of 8 are feasible.

G= 3/8 (no information about spin)

S_z = +1: $\psi_1 \sim \frac{1}{\sqrt{2}}(p\uparrow n\uparrow - n\uparrow p\uparrow)$

S_z = 0: $\psi_2 \sim \frac{1}{2}(p\uparrow n\downarrow + p\downarrow n\uparrow - n\uparrow p\downarrow - n\downarrow p\uparrow)$

S_z = -1: $\psi_3 \sim \frac{1}{\sqrt{2}}(p\downarrow n\downarrow - n\downarrow p\downarrow)$

$$\psi_4 \sim \frac{1}{2}(p\uparrow n\downarrow - p\downarrow n\uparrow - n\uparrow p\downarrow + n\downarrow p\uparrow)$$

$$\psi_5 \sim \frac{1}{\sqrt{2}}(p\uparrow n\uparrow + n\uparrow p\uparrow)$$

$$\psi_6 \sim \frac{1}{2}(p\uparrow n\downarrow + p\downarrow n\uparrow + n\uparrow p\downarrow + n\downarrow p\uparrow)$$

$$\psi_7 \sim \frac{1}{\sqrt{2}}(p\downarrow n\downarrow + n\downarrow p\downarrow)$$

$$\psi_8 \sim \frac{1}{2}(p\uparrow n\downarrow - p\downarrow n\uparrow + n\uparrow p\downarrow - n\downarrow p\uparrow)$$

$$p\uparrow \& n\uparrow \longrightarrow G' = 1/2(S_z = +1)$$

$$p\uparrow \& n\downarrow \longrightarrow G' = 1/4(S_z = 0)$$

$$p\downarrow \& n\uparrow \longrightarrow G' = 1/4(S_z = 0)$$

$$p\downarrow \& n\downarrow \longrightarrow G' = 1/2(S_z = -1)$$

$^{3}_{1}H$ & $^{3}_{2}He$ wave function

S=1/2 T=1/2

$$|^{3}_{1}H / ^{3}_{2}He\rangle \sim |spin\rangle|isospin\rangle \qquad S_\rho T_\lambda - S_\lambda T_\rho$$

S **T**

$$S=3/2\begin{cases}\uparrow\uparrow\uparrow \\ \frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow+\downarrow\uparrow\uparrow+\uparrow\downarrow\uparrow) \\ \frac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow+\uparrow\downarrow\downarrow+\downarrow\uparrow\downarrow) \\ \downarrow\downarrow\downarrow\end{cases} \qquad \begin{cases} ppp \\ \frac{1}{\sqrt{3}}(ppn+npp+pnp) \\ \frac{1}{\sqrt{3}}(nnp+pnn+npn) \\ nnn\end{cases} T=3/2$$

$$S=1/2\ \rho \begin{cases}\frac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow-\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow) \\ \frac{1}{\sqrt{6}}(\uparrow\downarrow\downarrow+\downarrow\uparrow\downarrow-2\downarrow\downarrow\uparrow)\end{cases} \qquad \begin{cases}\frac{1}{\sqrt{6}}(2ppn-pnp-npp) \\ \frac{1}{\sqrt{6}}(pnn+npn-2nnp)\end{cases} T=1/2$$

$$S=1/2\ \lambda \begin{cases}\frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow) \\ \frac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow-\downarrow\uparrow\downarrow)\end{cases} \qquad \begin{cases}\frac{1}{\sqrt{2}}(pnp-npp) \\ \frac{1}{\sqrt{2}}(pnn-npn)\end{cases} T=1/2$$

$$\{S(^{3}_{1}H)=1/2 \,\&\, S(^{3}_{2}He)=1/2\}$$

$$|^{3}_{2}He\rangle(S_z\uparrow)$$

$$\psi_1 \sim \frac{1}{\sqrt{6}}(p\uparrow n\uparrow p\downarrow - p\downarrow n\uparrow p\uparrow - n\uparrow p\uparrow p\downarrow + n\uparrow p\downarrow p\uparrow - p\uparrow p\downarrow n\uparrow + p\downarrow p\uparrow n\uparrow)$$

$$\psi_2 \sim \frac{1}{2}(p\uparrow n\uparrow p\downarrow + n\uparrow p\uparrow p\downarrow - p\uparrow p\downarrow n\uparrow - p\downarrow p\uparrow n\uparrow)$$

$$\psi_3 \sim \frac{1}{\sqrt{12}}(-p\uparrow n\uparrow p\downarrow - 2p\downarrow n\uparrow p\uparrow + n\uparrow p\uparrow p\downarrow + 2n\uparrow p\downarrow p\uparrow + p\uparrow p\downarrow n\uparrow - p\downarrow p\uparrow n\uparrow)$$

24 wave function(considering the spin-isospin and exchange of antisymmetric), 2 of 24 are feasible.

G= 1/12 (no information of spin)

$^{3}_{2}He$ G'

$$n\uparrow \& p\uparrow \& p\downarrow \longrightarrow 1/3(S_z=+1/2)$$

$$n\downarrow \& p\uparrow \& p\downarrow \longrightarrow 1/3(S_z=-1/2)$$

Similar for $^{3}_{1}H$