Communications

Theoretical Investigation on the Inverse Black Body Radiation Problem

NAN-XIAN CHEN AND GUANG-YIN LI

Abstract—**After reviewing previous works on the inverse black body radiation problem, a completely new approach is presented to solve the problem not only theoretically but also numerically. The first exact and concise expression for the solution is shown in this work based on the Fourier transform theory. Its existence, uniqueness, and stability are discussed in detail. The problem of convergence has been completely eliminated.**

I. Introduction

The inverse black body radiation problem has been given much attention in recent literature [1]–[8]. The problem is to determine the area temperature distribution $a(T)$ from the measured total radiation power spectrum $W(\nu)$ of a black body with temperature T, where ν is the frequency.

Bojarski proposed the first formulation for the problem in 1982 [1]. He presented a numerical solution by using the Laplace transform and iterative process. Subsequently, various authors provided different improvements based on his solution [2]–[8]. All these authors used the inverse Laplace transform, but none of them discussed the existence and uniqueness of the inverse Laplace transform. Also, none of them solved the convergence of the iterative procedure or of the series expansion. Lakhtakia and Lakhtakia had discussed the algorithm-independent check about this inverse problem, but they did not give any concrete examples [9].

Recently a numerical approach for obtaining stable solutions was given by Sun and Jaggard [10] based on Tinkhonov regularization [11], [12], but no general expression for the solution is shown directly and an unknown regularization parameter is to be chosen and adjusted in the approach.

A completely new method for solving the problem has been proposed [13]. In this work, a newly modified version with numerical results is presented in detail. An exact and concise expression for the general solution is given based on functional transforms and Fourier convolutions. This method eliminated the difficulties not only of existence and uniqueness, but also of convergence. In specific examples, this solution gives precise numerical results for a large range of temperature distributions, because the author uses the widely known and easily implemented fast Fourier transform (FFT) procedure.

II. Review of Previous Works

The power spectrum $P(\nu, T)$ radiated by unit area of a black body with absolute temperature T is given by Planck's law as

$$P(\nu, T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT} - 1} \tag{1}$$

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The authors are with the Institute of Applied Physics, Beijing University of Science and Technology, Beijing 100083, China.

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where h is Planck's constant, k is Boltzmann's constant, and c is the velocity of light.

If the area temperature distribution of the black body is $a(T)$, then total radiated power spectrum $W(\nu)$ is

$$W(\nu) = \frac{2h\nu^3}{c^2}\int_0^\infty \frac{a(T)\,dT}{e^{h\nu/kT} - 1}. \tag{2}$$

The inverse black body radiation problem to determine the area-temperature distribution $a(T)$ based on the measured total power spectrum $W(\nu)$ thus consists of solving the integral equation (2) for $a(T)$.

The solution of this problem was first presented by Bojarski. He introduced new variables "absolute coldness" u and "area-coldness distribution" $a(u)$ [1]

$$u = h/kT \tag{3}$$

$$a(u)\,du = -a(T)\,dT \tag{4}$$

then (2) can be written as

$$W(\nu) = \frac{2h\nu^3}{c^2}\int_0^\infty \frac{a(u)\,du}{e^{u\nu} - 1} \tag{5}$$

where $u\nu > 0$. The integral equation (5) can be rewritten by using a series expansion of the denominator of the integrand in (5) as

$$W(\nu) = \frac{2h\nu^3}{c^2}\int_0^\infty e^{-u\nu}\sum_{n=1}^\infty \frac{1}{n} a\left(\frac{u}{n}\right) \tag{6}$$

Let

$$f(u) = \sum_{n=1}^\infty \frac{1}{n} a\left(\frac{u}{n}\right) \tag{7}$$

and

$$g(\nu) = \frac{c^2}{2h}\frac{W(\nu)}{\nu^3}. \tag{8}$$

From (6)–(8), we see that $f(u)$ is the inverse Laplace transform of $g(\nu)$, i.e.,

$$f(u) = L^{-1}[g(\nu)]. \tag{9}$$

In fact, $f(u)$ is the solution in Wien approximation.

Bojarski used an iterative procedure, choosing $L^{-1}[g(\nu)]$ as the first iterative value, thus

$$a_{m+1}(u) = f(u) - \sum_{n=2}^\infty \frac{1}{n} a_m\left(\frac{\nu}{n}\right) \tag{10}$$

$$a(u) = \lim_{m\to\infty} a_m(u). \tag{11}$$

Kim and Jaggard [2] eliminated the iterative procedure and presented an explicit series expression for the solution. The final result is obtained as [15]

$$a(u) = \sum_{n=1}^\infty \frac{\mu(n)}{n} f\left(\frac{u}{n}\right) \tag{12}$$

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where $\mu(n)$ are Möbius functions in the theory of numbers. Hunter [4], and Hamid and Ragheb [3] considered the situation at microwave frequencies where the Rayleigh–Jeans approximation $h\nu \ll kT$ holds. Lakhtakia and Lakhtakia [6] discussed the case at submillimeter wavelengths, and it is equivalent to another Bojarski work [5].

All the methods mentioned above used the Laplace transform and the inverse Laplace transform, but did not verify the existence and uniqueness of the solution. The convergence of iteration or series expansion for these "universal solutions" was not discussed either. There exist some difficulties in both formulation and numerical computation.

Recently, Sun and Jaggard [10] have presented a numerical approach based on Tinkhonov regularization [11], [12]. A stable numerical solution was given if an unknown parameter was chosen adequately.

III. New Solution of the Inverse Problem

To solve the difficulties in the above-mentioned solutions for an inverse black body radiation problem, a completely new method was presented by Chen [13]. Here the modified version with specific calculated results is presented. Introducing the new variables as

$$e^x = h\nu/kT_0, \quad e^y = T/T_0 \tag{13}$$

where T_0 is an appropriate reference temperature, then (2) can be rewritten as

$$W(kT_0e^x/h) = \frac{2k^3T_0^4}{h^2c^2}e^{3x}\int_{-\infty}^{\infty}\frac{a(T_0e^y)e^y}{e^{e^{x-y}}-1}\,dy. \tag{14}$$

Let

$$G(x) = \frac{h^2c^2}{2k^3T_0^4}W\left(\frac{kT_0e^x}{h}\right)e^{-(2-\Delta)x}. \tag{15}$$

$$A(y) = e^{(2+\Delta)y}a(T_0e^y). \tag{16}$$

Hence

$$G(x) = \int_{-\infty}^{\infty}\phi(x-y)A(y)\,dy \tag{17}$$

where the integral kernel (with $\Delta = 0.5$, 1.0, 1.5, 2.0 in Fig. 1) is

$$\phi(u) = \frac{e^{(1+\Delta)u}}{e^{e^u}-1}. \tag{18}$$

The reference temperature T_0 can be estimated as centroid of area-temperature distribution in general. Therefore, the distribution of variable y is near to zero, which is convenient for the computation.

Since

$$\lim_{u\to\pm\infty}\phi(u) = 0, \qquad (0 < \Delta < 3) \tag{19}$$

and the energy of a real system is finite, $\phi(u)$ should be an integrable function for all the range of variable $u \in (-\infty, +\infty)$. Obviously, $A(y)$ and $G(x)$ are also integrable. Therefore, for all the $\phi(u)$, $A(y)$ and $G(x)$, one can carry out Fourier transforms. From (17), G is the convolution of ϕ and A, i.e.,

$$G(x) = \Phi(x)*A(x) \tag{20}$$

Therefore

$$F[A] = F[G]/F[\phi].$$

Finally the temperature distribution is given by

$$A = F^{-1}\{F[G]/F[\Phi]\}. \tag{21}$$

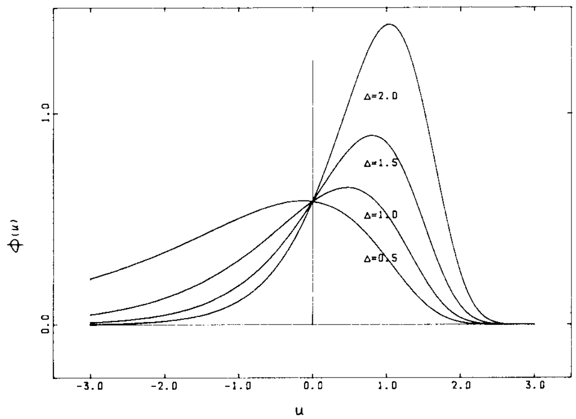

Fig. 1. Kernel $\phi(u)$ with different parameters as functions of u.

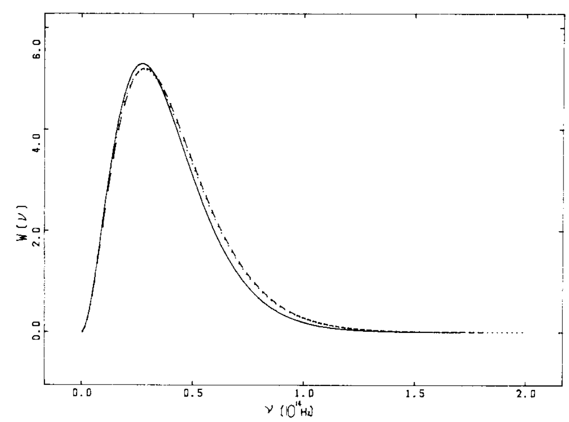

Fig. 2. Radiated power spectra related to the black bodies with temperature distributions of rectangle (solid line), Gaussian (dotted line), and triangle (dashed line).

Thus, the concise and exact form of the unknown temperature distribution can be obtained from the measurable power spectra.

On the other hand, to do a specific calculation for A or $a(T)$ based on (21), it is required that the nonzero spectrum of ϕ, i.e., $F[\phi]$, include the range of the nonzero part of $F[G]$. It is obvious that $F[G]$ is broadest when $A(x)$ is a delta-function. In this case, the width of $F[G]$ is the same as $F[\phi]$. This is why the range of nonzero; spectrum of $G(x)$ is always narrower than that of $\phi(x)$. In addition, $A(x)$ is of finite broadness since the temperature can never be either zero or infinity; thus the existence and uniqueness of this inverse problem is shown.

$\phi(u)$ is dependent on the parameter Δ. It is noticed that the smaller the Δ is, the wider the distribution of $\phi(u)$ is, and the narrower the $F[\phi(u)]$. Therefore, $\phi(u)$ with small Δ is convenient for the calculations shown later, where Δ is chosen as $\Delta = 1$.

IV. Calculated Results

Three kinds of power spectra $W(\nu)$ shown in Fig. 2 have been considered at first in our calculations. The corresponding temperature distributions are mainly in the range from 300K to 600K with the shapes of rectangle, Gaussian, and triangle, respectively. The frequency range of the spectra is about 5×10 in to 2×10^{14} in our calculations. The task of solving the inverse problem is to reconstruct the quite different area-temperature distributions $a(T)$ from

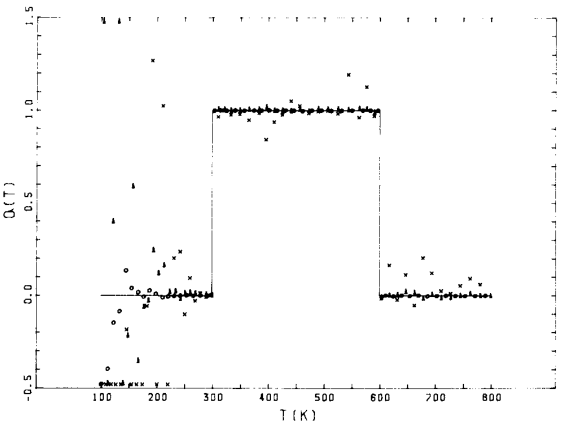

Fig. 3. The calculated temperature distributions along with ideal curve (rectangle, solid line). The corresponding noise levels are 1.0%(×), 0.1% (Δ), 0.01% (○), respectively.

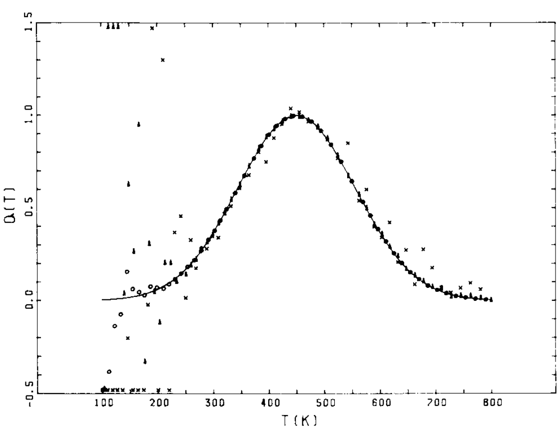

Fig. 4. Calculated temperature distributions along with ideal Gaussian curve.

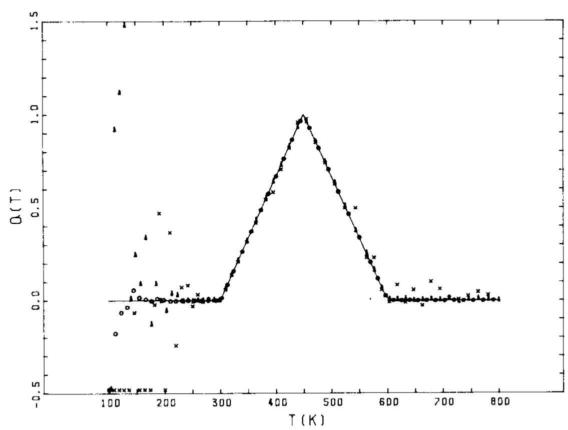

Fig. 5. Calculated temperature distributions along with ideal triangular curve.

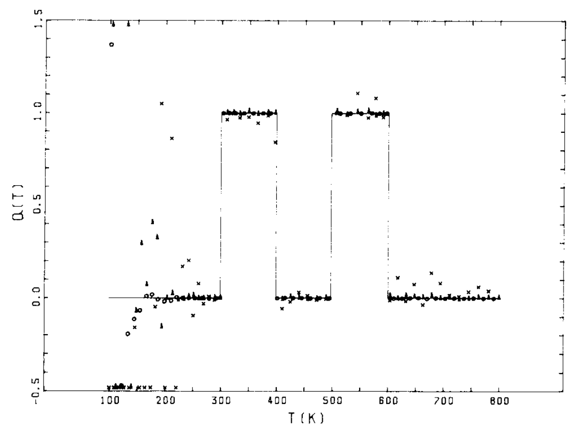

Fig. 6. Calculated temperature distributions along with ideal curve of double rectangles.

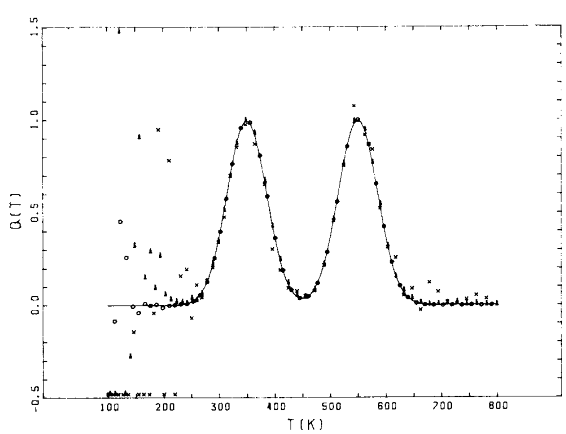

Fig. 7. Calculated temperature distributions along with ideal curve of double Gaussian.

these very similar power spectra $W(\nu)$. The calculations in this paper are based on (21), (15), and (16) introduced in Section III. Considering the error in a practical situation, a random noise with different levels was applied to the power spectrum $W(\nu)$. Fig. 3 shows the comparison of the ideal $a(T)$ values indicated by the solid lines and the calculated results for $a(T)$ with noise/signal ratio of 1% (x), 0.1% (Δ) and 0.01% (○) in $W(\nu)$. In the case of the calculated $a(T)$ value exceeding the frame of the diagram, the corresponding point has been moved to the boundary of the figure. Obviously the overall agreement between calculated and ideal values is excellent for the cases of noise level below 0.1%. The apparent oscillations in the low temperature region indicate the intrinsic instability of the physical problem. In fact, the radiated power from a low temperature surface is low (shown in Fig. 9), thus it is very difficult to distinguish the power spectra from different low temperature sources. Figs. 3 and 4 show the calculated results for the cases of Gaussian and triangular distributions along with the ideal curves. All the marks used are the same as in Fig. 2, and the agreement is satisfactory again. In the above mentioned cases, the kernel parameter Δ and the reference temperature T_0 are chosen as 1.0 and 450K, respectively.

Comparing with most of the previous works [1]–[10], the calculation in this work is adequate for all wave bands instead of only for long wave or short wave limit. Since the calculation involved is mainly the FFT operation, the procedure is carried out only in a very short time by using an IBM PC/XT machine with double precision.

Figs. 6–8 show the results for temperature distributions with double peaks. The calculated results, again, agree with the ideal values very well provided that the error in $W(\nu)$ is small. When one uses the power spectra $W(\nu)$ without noise, the error of the resultant $a(T)$

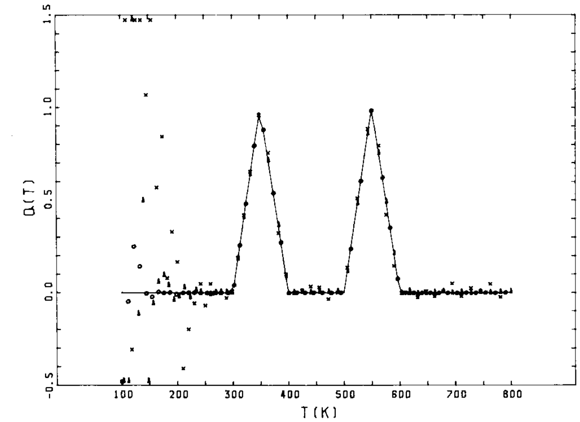

Fig. 8. Calculated temperature distributions along with ideal curve of double triangles.

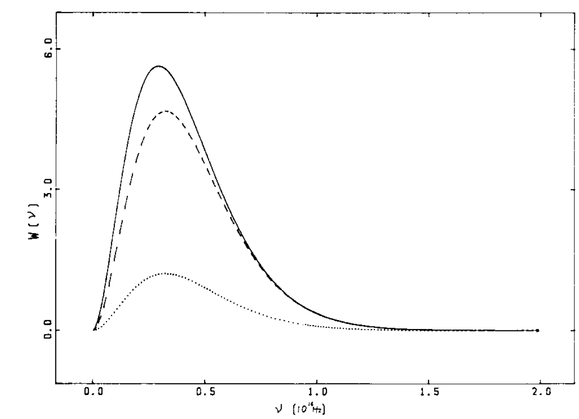

Fig. 9. Power spectra $W(\nu)$ corresponding to the case shown in Fig. 8. Solid line is from the total temperature distribution, dotted line from the low temperature triangle, and dashed line from the high temperature triangle.

is less than 10^{-11}. There are no limitations to the temperature range and to the shape of temperature distribution in our calculations.

V. Conclusion and Discussion

The method for solving the inverse problem in this paper is different in essence from previous methods. The problems of existence, uniqueness, and convergence are solved completely. The stability is checked by addition of noise and the instability in low temperature region is explained. Also, the quite simple FFT only is used in the numerical calculation. Not surprisingly, it is very convenient, and can be used without frequency limitation. Our method is suitable not only for the inverse black body radiation problem, but also for a large number of Fredholm integral equations of the first kind in physics [14].

It is still necessary to consider the correction caused by a real body, and to determine the local temperature all over the surface. Also, one can measure $W(\nu)$ only for some $[\nu_1, \nu_2]$ and $W(\nu)$ could be nonzero for a wide range. These problems remain for further consideration.

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The Surface Impedance of Metallic Objects: Rigorous Calculations for Imperfectly Conducting Diffraction Gratings

RICARDO A. DEPINE AND VERA L. BRUDNY

Abstract—**An accurate computation of the surface impedance of highly conducting metallic gratings is presented. A rigorous electromagnetic formalism is used for solving for the fields at the grating boundary. The influence of the local curvature of the grating profile, the refractive index of the metal, and the angle of incidence are studied. Particular attention is given to the comparison between the efficiencies obtained using the rigorous method and those obtained assuming that the surface impedance is a constant.**

Introduction

We shall consider the problem of the diffraction of a plane wave at a periodic boundary S separating two media. Usually the scattering problem is solved by using the pair of Maxwell boundary conditions

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The authors are with the Laboratorio de Optica, Departamento de Fisica, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón 1, 1428 Buenos Aires, Argentina.
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