

Test of quantum effects of spatial noncommutativity using modified electron momentum spectroscopy

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(Received 24 September 2008; published 24 November 2008)

The possibility of testing spatial noncommutativity by current experiments on normal quantum scales is investigated. In the case of both position-position and momentum-momentum noncommuting the spectra of ions in crossed electric and magnetic fields are studied in the formalism of noncommutative quantum mechanics. In a limit of the kinetic energy approaching its lowest eigenvalue, this system possesses nontrivial dynamics. Signals of spatial noncommutativity in the angular momentum are revealed. They are within limits of the measurable accuracy of current experiments. An experimental test of the predictions using a modified electron momentum spectrum is suggested. The related experimental sensitivity and subtle points are discussed. The results are the first step on a realizable way towards a conclusive test of spatial noncommutativity.

DOI: [10.1103/PhysRevD.78.105021](https://doi.org/10.1103/PhysRevD.78.105021)

PACS numbers: 03.65.-w

Quantum theory in noncommutative space [1–9] presents an attractive possibility as a candidate in the present round in hinting at new physics. In the low energy aspect, quantum mechanics in noncommutative space (NCQM) [10–18] have been studied in detail. But testing their predictions requires experiments near the Planck scale which are unrealizable, and/or their modifications to normal quantum theory depending on vanishingly small noncommutative parameters, which are outside the limits of measurable accuracy of current experiments. It seems that noncommutative quantum theory escapes measurement on normal quantum scales.

The possibility of testing spatial noncommutativity by current experiments on normal quantum scales was investigated, and two proposals [18] using Rydberg atoms and Chern-Simons processes are suggested. The two proposals revealed that in a limit of diminishing the magnetic field to zero the vanishingly small noncommutative parameters usually present in predictions derived from spatial noncommutativity actually cancel out in the angular momentum, so that the lowest angular momentum turns out to be $\hbar/4$. This provides a conclusive test of spatial noncommutativity, i.e., a positive experimental result shows an evidence of spatial noncommutativity and a negative one draws a conclusive preclusion of spatial noncommutativity. In practice the magnetic field, however, can only reach some finite value limited by the level of shielding the background magnetic fields. In order to meet the condition of a cancellation between the vanishingly small noncommutative parameters present in the angular momentum

derived from spatial noncommutativity, the magnetic field must be decreased to a level of some orders less than the effective intrinsic magnetic field B_η originated from spatial noncommutativity (see below). The field control at that level seems close to impossible in the foreseeable future.

According to the present level of shielding the background magnetic fields, this paper explores a realizable way for testing noncommutative quantum effects on normal quantum scale. In the case of both position-position and momentum-momentum noncommuting the spectra of ions trapped in crossed electric and magnetic fields are investigated. In a limit of the kinetic energy approaching its lowest eigenvalue, this system possesses nontrivial dynamics. The corresponding constraints are analyzed. Signals of spatial noncommutativity in the angular momentum are revealed. They are within limits of the measurable accuracy of current experiments. An experimental test of the predictions using, similar to electron momentum spectroscopy (EMS) [19], a modified EMS is suggested, and the related experimental sensitivity and the subtle points are discussed. It is the first step on a realizable way towards a conclusive test of spatial noncommutativity.

We consider an ion of mass μ and charge $q(>0)$ trapped in a uniform magnetic field \mathbf{B} aligned along the x_3 axis and an electrostatic potential [20]

$$V_{\text{eff}} = \frac{1}{2}\mu[\omega_\rho^2(x_1^2 + x_2^2) + \omega_z^2x_3^2] \quad (1)$$

where ω_ρ^2 and ω_z^2 are frequencies, respectively, in the (x_1, x_2) plane and z direction. The vector potential A_i of \mathbf{B} is chosen as $A_i = -B\epsilon_{ij}x_j/2$, $A_3 = 0$, ($i, j = 1, 2$). The Hamiltonian H of the trapped ion can be decomposed into a one-dimensional harmonic Hamiltonian H_z in the z di-

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rejection and a two-dimensional Hamiltonian H_2 in the (x_1, x_2) plane: $H = (p_i - qA_i/c)^2/2\mu + V_{\text{eff}} = H_2 + H_z$, $H_z = p_3^2/2\mu + \mu\omega_z^2 x_3^2/2$, and H_2 is (henceforth the summation convention is used)

$$\begin{aligned} H_2 &= \frac{1}{2\mu} \left(p_i + \frac{1}{2} \mu \omega_c \epsilon_{ij} x_j \right)^2 + \frac{1}{2} \kappa x_i^2 \\ &= \frac{1}{2\mu} p_i^2 + \frac{1}{2} \omega_c \epsilon_{ij} p_i x_j + \frac{1}{2} \mu \omega_p^2 x_i^2, \end{aligned} \quad (2)$$

where $\kappa = \mu\omega_p^2$, $\omega_c = qB/\mu c$ (the cyclotron frequency), and $\omega_p = (\omega_\rho^2 + \omega_c^2/4)^{1/2}$.

If NCQM is a realistic physics, low energy quantum phenomena should be reformulated in the formalism of NCQM. We consider the case of both position-position noncommutativity (position-time noncommutativity is not considered) and momentum-momentum noncommutativity. The consistent deformed Heisenberg-Weyl algebra [18] is

$$\begin{aligned} [\hat{x}_I, \hat{x}_J] &= i\xi^2 \theta_{IJ}, & [\hat{x}_I, \hat{p}_J] &= i\hbar \delta_{IJ}, \\ [\hat{p}_I, \hat{p}_J] &= i\xi^2 \eta_{IJ}, & (I, J &= 1, 2, 3) \end{aligned}$$

where θ_{IJ} and η_{IJ} are the antisymmetric constant parameters, independent of the position and momentum. We define $\theta_{IJ} = \epsilon_{IJK} \theta_K$, where ϵ_{IJK} is a three-dimensional antisymmetric unit tensor. We put $\theta_3 = \theta$ and the rest of the θ components to zero (which can be done by a rotation of coordinates), then we have $\theta_{ij} = \epsilon_{ij} \theta$ ($i, j = 1, 2$), where $\epsilon_{ij} = \epsilon_{ij3}$ is a two-dimensional antisymmetric unit tensor with $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$. Similarly, we have $\eta_{ij} = \epsilon_{ij} \eta$. Thus we obtain the following two-dimensional deformed Heisenberg-Weyl algebra

$$\begin{aligned} [\hat{x}_i, \hat{x}_j] &= i\xi^2 \theta_{ij}, & [\hat{x}_i, \hat{p}_j] &= i\hbar \delta_{ij}, \\ [\hat{p}_i, \hat{p}_j] &= i\xi^2 \eta_{ij}, & (i, j &= 1, 2). \end{aligned} \quad (3)$$

Here we consider the noncommutativity of the intrinsic canonical momentum. It means that the parameter η , like the parameter θ , should be extremely small. This is guaranteed by a direct proportionality between them [see Eq. (39) below]. In Eqs. (3) the scaling factor ξ is $\xi = (1 + \theta\eta/4\hbar^2)^{-1/2}$.

The deformed Heisenberg-Weyl algebra (3) can be realized by undeformed variables x_i and p_i as follows

$$\hat{x}_i = \xi \left(x_i - \frac{1}{2\hbar} \theta \epsilon_{ij} p_j \right), \quad \hat{p}_i = \xi \left(p_i + \frac{1}{2\hbar} \eta \epsilon_{ij} x_j \right), \quad (4)$$

where x_i and p_i satisfy the undeformed Heisenberg-Weyl algebra $[x_i, x_j] = [p_i, p_j] = 0$, $[x_i, p_j] = i\hbar \delta_{ij}$. It should be emphasized that for the case of both position-position and momentum-momentum noncommuting, the scaling factor ξ in Eqs. (3) and (4) guarantees consistency of the framework, and plays an essential role in dynamics.

The deformed Hamiltonians \hat{H}_2 and \hat{H}_z in noncommutative space can be obtained by reformulating the corresponding undeformed ones in commutative space in terms of deformed canonical variables \hat{x}_i and \hat{p}_i . Because of $\hat{x}_3 = x_3$ and $\hat{p}_3 = p_3$, the deformed $\hat{H}_z(\hat{x}_3, \hat{p}_3)$ is the same as the undeformed one, $\hat{H}_z(\hat{x}_3, \hat{p}_3) = H_z(x_3, p_3)$.

The deformed $\hat{H}_2(\hat{x}, \hat{p})$, using Eqs. (4), can be further represented by undeformed variables x_i and p_i as

$$\begin{aligned} \hat{H}_2(\hat{x}, \hat{p}) &= \frac{1}{2M} \left(p_i + \frac{1}{2} G \epsilon_{ij} x_j \right)^2 + \frac{1}{2} K x_i^2 \\ &= \frac{1}{2M} p_i^2 + \frac{1}{2M} G \epsilon_{ij} p_i x_j + \frac{1}{2} M \Omega_p^2 x_i^2, \end{aligned} \quad (5)$$

where the effective parameters M , G , Ω_p , and K are defined as

$$\begin{aligned} 1/2M &\equiv \xi^2 (c_1^2/2\mu + \kappa\theta^2/8\hbar^2), \\ G/2M &\equiv \xi^2 (c_1 c_2/\mu + \kappa\theta/2\hbar), \\ M\Omega_p^2 &\equiv \xi^2 (c_2^2/\mu + \kappa), \\ K &\equiv M\Omega_p^2 - G^2/4M, \end{aligned} \quad (6)$$

and $c_1 = 1 + \mu\omega_c\theta/4\hbar$, $c_2 = \mu\omega_c/2 + \eta/2\hbar$.

The deformed Hamiltonian (2) and the equivalent one of Eq. (5) possess a rotational symmetry in the (x_1, x_2) plane. The z component of the orbital angular momentum is a conserved observable.

In order to explore new features of such a system, we need to investigate a Chern-Simons term $\hat{J}_z = \epsilon_{ij} \hat{x}_i \hat{p}_j$. From the NCQM algebra (3) we obtain commutation relations between \hat{J}_z and \hat{x}_i , \hat{p}_i : $[\hat{J}_z, \hat{x}_i] = i\epsilon_{ij} \hat{x}_j + i\xi^2 \theta \hat{p}_i$, $[\hat{J}_z, \hat{p}_i] = i\epsilon_{ij} \hat{p}_j - i\xi^2 \eta \hat{x}_i$. From the above commutation relations we conclude that \hat{J}_z plays approximately the role of the generator of rotations at the deformed level. Using Eqs. (4), we can further represent \hat{J}_z by undeformed variables x_i and p_i as

$$\begin{aligned} \hat{J}_z &= \epsilon_{ij} x_i p_j - \frac{1}{2\hbar} \xi^2 (\theta p_i p_i + \eta x_i x_i) \\ &= J_z - \frac{1}{2\hbar} \xi^2 (\theta p_i p_i + \eta x_i x_i), \end{aligned} \quad (7)$$

where $J_z = \epsilon_{ij} x_i p_j$ is the z component of the orbital angular momentum in commutative space. \hat{J}_z and \hat{H}_2 commute each other. They have common eigenstates.

We investigate an interesting case: dynamics of this system in the limit of the mechanical kinetic energy approaching its lowest eigenvalue. For discussing this limit it is convenient to work in the Lagrangian formalism. The Lagrangian corresponding to the Hamiltonian \hat{H}_2 in Eq. (5) is

$$\hat{L} = \frac{1}{2} M \dot{x}_i \dot{x}_i + \frac{1}{2} \epsilon_{ij} \dot{x}_i x_j - \frac{1}{2} K x_i x_i. \quad (8)$$

The mechanical kinetic energy $\hat{H}_k = M \dot{x}_i \dot{x}_i / 2$ can be rewritten as

$$\hat{H}_k = \frac{1}{2M}(K_1^2 + K_2^2) \quad (9)$$

where

$$K_i \equiv p_i + \frac{1}{2}G\epsilon_{ij}x_j \quad (10)$$

are the mechanical momentum corresponding to the vector potentials A_i . They satisfy the commutation relation

$$[K_i, K_j] = i\hbar G\epsilon_{ij}. \quad (11)$$

In Eq. (10) $p_i = \partial/\partial x_i$ are the canonical momentum. They satisfy the commutation relation $[p_i, p_j] = 0$.

We define canonical variables $Q = K_1/G$ and $\Pi = K_2$, which satisfy

$$[Q, \Pi] = i\hbar\delta_{ij}. \quad (12)$$

The kinetic energy \hat{H}_k in Eq. (9) is rewritten as a Hamiltonian of a harmonic oscillator

$$\hat{H}_k = \frac{1}{2M}\Pi^2 + \frac{1}{2}M\omega_0^2Q^2, \quad (13)$$

where the effective frequency $\omega_0 \equiv G/M$. The eigenvalues of \hat{H}_k are

$$\hat{\mathcal{E}}_{k,n} = \hbar\omega_0(n + \frac{1}{2}), \quad (n = 0, 1, 2, \dots). \quad (14)$$

Its lowest one is

$$\hat{\mathcal{E}}_{k,0} = \frac{\hbar G}{2M}. \quad (15)$$

In the limit of $\hat{H}_k \rightarrow \hat{\mathcal{E}}_{k,0}$, the Hamiltonian \hat{H}_2 in Eq. (5) reduces to

$$\hat{H}_0 = \frac{1}{2}Kx_ix_i + \hat{\mathcal{E}}_{k,0}. \quad (16)$$

The Lagrangian corresponding to \hat{H}_0 is

$$\hat{L}_0 = \frac{1}{2}G\epsilon_{ij}\dot{x}_i x_j - \frac{1}{2}Kx_ix_i - \hat{\mathcal{E}}_{k,0}. \quad (17)$$

In the following, we demonstrate that the reduced system (\hat{H}_0, \hat{L}_0) has nontrivial dynamics. The canonical momenta

$$p_i = \frac{\partial \hat{L}_0}{\partial \dot{x}_i} = \frac{1}{2}G\epsilon_{ij}x_j. \quad (18)$$

The Hamiltonian \hat{H}'_0 obtained from \hat{L}_0 is $\hat{H}'_0 = p_i \dot{x}_i - \hat{L}_0 = Kx_ix_i/2 + \hat{\mathcal{E}}_{k,0}$, which is just \hat{H}_0 .

The canonical momenta p_i in Eq. (18) does not determine velocities \dot{x}_i as functions of p and x which indicates that \hat{L}_0 is singular, but gives relations among p and x . Such relations are primary constraints [21,22]

$$\varphi_i(x, p) = p_i + \frac{1}{2}G\epsilon_{ij}x_j = 0. \quad (19)$$

The physical meaning of Eq. (19) is that it expresses the dependence of degrees of freedom among p and x . The constraints (19) should be carefully treated.

The Hamiltonian equation of \hat{H}_0 in Eq. (16) gives $\dot{x}_i = \partial \hat{H}_0 / \partial p_i = 0$. But the \hat{L}_0 in Eq. (17) has nonvanishing \dot{x}_i . This needs to be clarified. The Hamiltonian equations of such a singular (constrained) system are not unique. Because of the constraints $\varphi_i(x, p) = 0$ of Eq. (19), \hat{H}_0 plus any linear combination of φ_i is also a Hamiltonian of the system, i.e., the \hat{H}_0 can be replaced by $\hat{H}_0(x, p) + \lambda_i \varphi_i(x, p)$ where the Lagrange multiplier λ_i may be a function of x and p . The Hamiltonian equations, including the contributions of $\delta(\lambda_i(x, p)\varphi_i(x, p))$, read

$$\dot{p}_i = -\frac{\partial \hat{H}_0}{\partial x_i} - \lambda_k \frac{\partial \varphi_k}{\partial x_i}, \quad \dot{x}_i = \frac{\partial \hat{H}_0}{\partial p_i} + \lambda_k \frac{\partial \varphi_k}{\partial p_i}. \quad (20)$$

From $\partial \hat{H}_0 / \partial p_i = 0$ and $\partial \varphi_k / \partial p_i = \delta_{ki}$, it follows that the above second equation reduces to

$$\dot{x}_i = \lambda_i. \quad (21)$$

In this example the Lagrange multiplier λ_i is just the velocity \dot{x}_i .

The Poisson brackets of the constraints are

$$C_{ij} = \{\varphi_i, \varphi_j\}_P = G\epsilon_{ij}. \quad (22)$$

C_{ij} defined in Eq. (22) are elements of the constraint matrix \mathcal{C} . Elements of its inverse matrix \mathcal{C}^{-1} are $(\mathcal{C}^{-1})_{ij} = -\epsilon_{ij}/G$. The corresponding Dirac brackets of the canonical variables x_i and p_j can be determined,

$$\{x_i, p_j\}_D = \frac{1}{2}\delta_{ij}, \quad \{x_1, x_2\}_D = -\frac{1}{G}, \quad \{p_1, p_2\}_D = -\frac{G}{4}. \quad (23)$$

The Dirac brackets of φ_i with any variables x_i and p_j are zero so that the constraints (19) are strong conditions. It can be used to eliminate dependent variables. If we select x_1 and p_1 as independent variables, from the constraints (19) we obtain $x_2 = -2p_1/G$, $p_2 = Gx_1/2$. Introducing new canonical variables $x = \sqrt{2}x_1$ and $p = \sqrt{2}p_1$, we have $\{x, p\}_D = 1$. The corresponding quantum commutation relation is $[x, p] = i\hbar$.

The reduced system (\hat{H}_0, \hat{L}_0) can be solved as follows. We define, respectively, the following effective mass and frequency

$$\hat{\mu}^* \equiv \frac{G^2}{2K}, \quad \hat{\omega}^* \equiv \frac{K}{G}, \quad (24)$$

then the Hamiltonian \hat{H}_0 reduces to

$$\hat{H}_0^* = \frac{1}{2\hat{\mu}^*}p^2 + \frac{1}{2}\hat{\mu}^*\hat{\omega}^{*2}x^2 + \hat{\mathcal{E}}_{k,0}. \quad (25)$$

Equation (25) shows that the following annihilation and creation operators can be introduced

$$\begin{aligned}\hat{A}^* &= \sqrt{\frac{\hat{\mu}^* \hat{\omega}^*}{2\hbar}} x + i \sqrt{\frac{1}{2\hbar \hat{\mu}^* \hat{\omega}^*}} p, \\ \hat{A}^{*\dagger} &= \sqrt{\frac{\hat{\mu}^* \hat{\omega}^*}{2\hbar}} x - i \sqrt{\frac{1}{2\hbar \hat{\mu}^* \hat{\omega}^*}} p.\end{aligned}\quad (26)$$

They satisfies $[\hat{A}^*, \hat{A}^{*\dagger}] = 1$. The eigenvalues of the number operator $\hat{N}^* = \hat{A}^{*\dagger} \hat{A}^*$ is $n = 0, 1, 2, \dots$. The Hamiltonian \hat{H}_0^* reads

$$\hat{H}_0^* = \hbar \hat{\omega}^* \left(\hat{A}^{*\dagger} \hat{A}^* + \frac{1}{2} \right) + \hat{\mathcal{E}}_{k,0}. \quad (27)$$

Similarly, in the limit of $\hat{H}_k \rightarrow \hat{\mathcal{E}}_{k,0}$, the Chern-Simons term \hat{J}_z in Eq. (7) reduce to

$$\hat{J}_z^* = \hbar \hat{\mathcal{J}}^* \left(\hat{A}^{*\dagger} \hat{A}^* + \frac{1}{2} \right), \quad \hat{\mathcal{J}}^* = 1 - \xi^2 \left(\frac{G\theta}{4\hbar} + \frac{\eta}{G\hbar} \right). \quad (28)$$

The eigenvalues of \hat{J}_z^* are $\hat{\mathcal{J}}_n^* = \hbar \hat{\mathcal{J}}^* (n + 1/2)$. Its lowest one is

$$\hat{\mathcal{J}}_0^* = \frac{1}{2} \hbar \hat{\mathcal{J}}^*. \quad (29)$$

In the present case of both position-position and momentum-momentum noncommuting, the second equation of Eq. (4), up to the first order of θ and η , can be rewritten as $\hat{p}_i = p_i + \eta \epsilon_{ij} x_j / 2\hbar = p_i + q B_\eta \epsilon_{ij} x_j / 2c$. It indicates that there is an effective intrinsic magnetic field B_η originated from spatial noncommutativity,

$$B_\eta = \frac{c\eta}{q\hbar}. \quad (30)$$

Because B_η should be vanishingly small, only the external magnetic field B decreasing to a level of closing to B_η , the small quantum effects of B_η from spatial noncommutativity are manifested obviously.

In the following we demonstrate that, because of the effective intrinsic magnetic field B_η , in a further limiting process of diminishing the external magnetic field B to zero the survived system also has nontrivial dynamics. In this limit the frequency ω_p reduces to $\omega_p = \omega_\rho$. Up to the first order of θ and η , we have $\xi = 1$. The effective parameters M , G , Ω_p , and K reduce, respectively, to \tilde{M} , \tilde{G} , $\tilde{\Omega}_p$, and \tilde{K} , which are defined by

$$\begin{aligned}\tilde{M} &\equiv \left(\frac{1}{\mu} + \frac{\mu \omega_\rho^2 \theta^2}{4\hbar^2} \right)^{-1} = \mu, \\ \tilde{G}\hbar &\equiv \mu^2 \omega_\rho^2 \theta + \eta \\ \tilde{\Omega}_p^2 &\equiv \omega_\rho^2 + \frac{\eta^2}{4\mu^2 \hbar^2} = \omega_\rho^2, \\ \tilde{K} &\equiv \tilde{M} \tilde{\Omega}_p^2 - \frac{\tilde{G}^2}{4\tilde{M}} = \mu \omega_\rho^2.\end{aligned}\quad (31)$$

We define the following effective mass and frequency

$$\tilde{\mu} \equiv \frac{\tilde{G}^2}{2\tilde{K}}, \quad \tilde{\omega} \equiv \frac{\tilde{K}}{\tilde{G}}, \quad (32)$$

and the annihilation and creation operators

$$\begin{aligned}\tilde{A} &= \sqrt{\frac{\tilde{\mu} \tilde{\omega}}{2\hbar}} x + i \sqrt{\frac{1}{2\hbar \tilde{\mu} \tilde{\omega}}} p, \\ \tilde{A}^\dagger &= \sqrt{\frac{\tilde{\mu} \tilde{\omega}}{2\hbar}} x - i \sqrt{\frac{1}{2\hbar \tilde{\mu} \tilde{\omega}}} p\end{aligned}\quad (33)$$

\tilde{A} and \tilde{A}^\dagger satisfies $[\tilde{A}, \tilde{A}^\dagger] = 1$. The eigenvalues of the number operator $\tilde{N} = \tilde{A}^\dagger \tilde{A}$ is $\tilde{n} = 0, 1, 2, \dots$. Then, up to the first order of θ and η , the \hat{H}_0^* and \hat{J}_z^* reduce, respectively, to the following \tilde{H}_0 and \tilde{J}_z :

$$\begin{aligned}\tilde{H}_0 &= \hbar \tilde{\omega} \left(\tilde{A}^\dagger \tilde{A} + \frac{1}{2} \right), \quad \tilde{J}_z = \hbar \tilde{\mathcal{J}} \left(\tilde{A}^\dagger \tilde{A} + \frac{1}{2} \right), \\ \tilde{\mathcal{J}} &= 1 - \frac{\eta}{\tilde{G}\hbar}.\end{aligned}\quad (34)$$

The eigenvalues of \tilde{H}_0 and \tilde{J}_z are, respectively,

$$\tilde{E}_n = \hbar \tilde{\omega} \left(\tilde{n} + \frac{1}{2} \right), \quad \tilde{J}_n = \hbar \tilde{\mathcal{J}} \left(\tilde{n} + \frac{1}{2} \right). \quad (35)$$

The term $\eta/\tilde{G}\hbar$ of $\tilde{\mathcal{J}}$ in Eq. (34) reads $\eta/\tilde{G}\hbar = 1/[1 + \mu^2 \omega_\rho^2 / (\eta/\theta)]$. Where η/θ is a positive finite constant of dimension mass⁻² time².

In the context of nonrelativistic quantum mechanics this can be elucidated from conditions of guaranteeing the deformed bosonic algebra in the case of both position-position and momentum-momentum noncommuting. The general representations of deformed annihilation and creation operators \hat{a}_i and \hat{a}_i^\dagger at the deformed level are determined by the deformed Heisenberg-Weyl algebra (3) and the deformed bosonic algebra

$$[\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = 1, \quad [\hat{a}_i, \hat{a}_j] = 0. \quad (36)$$

They are [18]

$$\begin{aligned}\hat{a}_i &= \sqrt{\frac{1}{2\hbar} \sqrt{\frac{\eta}{\theta}}} \left(\hat{x}_i + \frac{i}{\sqrt{\eta/\theta}} \hat{p}_i \right), \\ \hat{a}_i^\dagger &= \sqrt{\frac{1}{2\hbar} \sqrt{\frac{\eta}{\theta}}} \left(\hat{x}_i - \frac{i}{\sqrt{\eta/\theta}} \hat{p}_i \right).\end{aligned}\quad (37)$$

In the limits $\theta, \eta \rightarrow 0$, and η/θ keeping finite, the deformed annihilation operator \hat{a}_i should reduce to the undeformed a_i in commutative space. In the context of nonrelativistic quantum mechanics, the general representations of undeformed annihilation and creation operators a_i and a_i^\dagger are determined by the undeformed Heisenberg-Weyl algebra and the undeformed bosonic algebra $[a_i, a_j^\dagger] = \delta_{ij}$, $[a_i, a_j] = 0$. They read

$$a_i = \sqrt{\frac{1}{2c\hbar}}(x_i + icp_i), \quad a_i^\dagger = \sqrt{\frac{1}{2c\hbar}}(x_i - icp_i), \quad (38)$$

where c is a positive constant. In the limits $\theta, \eta \rightarrow 0$, and η/θ keeping finite Eq. (37) should reduce to Eq. (38). It follows that the factor η/θ in Eq. (37) should equal c^{-2} , that is,

$$\eta = c^{-2}\theta. \quad (39)$$

Both noncommutative parameters η and θ should be extremely small, because modifications to the normal quantum mechanics originated from spatial noncommutativity should be vanishingly small. This is guaranteed by Eq. (39).

The undeformed Heisenberg-Weyl algebra shows that the equation $[a_i, a_j] = 0$ is automatically satisfied, thus there is not constraint on the constant c . Up to now, how to fix this constant from fundamental principles is an open issue.

From Eqs. (35) and (39) the dominant values of the lowest eigenvalue $\tilde{\mathcal{J}}_0$ and the interval $\Delta\tilde{\mathcal{J}}_n \equiv \tilde{\mathcal{J}}_{n+1} - \tilde{\mathcal{J}}_n$ of $\tilde{\mathcal{J}}_z$ take, respectively,

$$\tilde{\mathcal{J}}_0 = \frac{\hbar}{2} \frac{1}{1 + 1/c^2 \mu^2 \omega_p^2}, \quad \Delta\tilde{\mathcal{J}}_0 = \hbar \frac{1}{1 + 1/c^2 \mu^2 \omega_p^2}. \quad (40)$$

Comparing with the corresponding results $\mathcal{J}_0 = \hbar/2$ and $\Delta\mathcal{J}_n = \hbar/2$ in commutative space, Eq. (40) reveals that in noncommutative space $\tilde{\mathcal{J}}_0 < \hbar/2$ and $\Delta\tilde{\mathcal{J}}_n < \hbar/2$. The result $\tilde{\mathcal{J}}_0 < \hbar/2$ is a clear signal of spatial noncommutativity.

Towards a test of spatial noncommutativity using modified EMS—EMS [19] is used in atomic and molecular physics to obtain unique information about the motion and correlation of valence electrons in atoms, molecules, and their ions. EMS involves the measurement of the relative differential cross section for the $(e, 2e)$ reaction on an atom or molecule as a function of the electron separation energy and the momenta of observed electrons. A calculation of the differential cross section requires a knowledge of the target and ion wave functions. The reaction can be considered as a measurement of properties of these wave functions if it is well understood that the cross section can be calculated within experimental error. The measured and calculated momentum distributions are different for different angular momentum states of the electrons in the target, e.g., for the s state the maximum of the momentum profile appears at the region of the vanishing momentum, but for the p state the minimum appears at the same region.

In modified EMS we study the $(I, 2I)$ reaction where I means an ion. We take the incidental ion as the same type as the target (trapped) one, and measure the relative differential cross section of the two out-going ions as a

function of the ion separation energy and the momenta of observed ions. Here information of the angular momentum is, different from EMS, for the whole target ion, not for an electron of the target ion. For our purpose the initial state of modified EMS is taken as $|i\rangle = |\alpha\rangle|\chi^{(+)}(\vec{k}_0)\rangle$, where $|\chi^{(+)}(\vec{k}_0)\rangle$ is a distorted wave of the incident ion and $|\alpha\rangle$ is an angular momentum eigenstate of the target ion. Information of the angular momentum of the trapped ion is included in the wave function of the initial state. As an analog of EMS, the modified EMS $(I, 2I)$ reaction can differentiate between wave functions of the trapped ions with different angular momenta from the measured momentum distributions of out-going ions. For normal quantum mechanics in commutative space, $\theta = \eta = 0$, in limits of $H_k \rightarrow \mathcal{E}_{k,0}$ and subsequent diminishing the magnetic field B to zero the effective parameter $\tilde{G} = 0$, so the effective frequency $\tilde{\omega}$ and the annihilation operator \tilde{A} in Eq. (34) cannot be defined. Using angular momentum wave functions in momentum space for both normal quantum mechanics and NCQM, calculating the differential cross sections of $(I, 2I)$ reaction, and comparing theoretical results with the measured ones, we are able to conclusively determine whether space is noncommutative, i.e., a positive experimental result shows evidence of spatial noncommutativity and a negative one draws a conclusive preclusion of spatial noncommutativity.

The existing upper bounds of θ and η are, respectively, $\theta/(\hbar c)^2 \leq (10 \text{ TeV})^{-2}$ [12] and $|\sqrt{\eta}| \leq 1 \mu\text{eV}/c$ [17]. From this upper bound of η the effective intrinsic magnetic field

$$B_\eta = \frac{c\eta}{q\hbar} \leq 10^{-14} \text{ T}. \quad (41)$$

In order to meet the condition of deriving Eqs. (40), the magnetic field B must be decreased to a level of some orders less than B_η . The diminishing level of B is determined by the level of shielding the background magnetic fields. The field control at that level seems close to impossible in the foreseeable future.

Recently, controlling a field at the 10^{-9} T level was realized using magnetic shields of two thick mu-metal cylinders [23]. It may relax the field control further to the challenging, but achievable 10^{-12} T level. It is interesting to consider the case of diminishing the external magnetic field B to the level of $\sim 10^{-9}$ T (10^{-12} T). Using laser trapping and cooling [24], the limit of $\hat{H}_k \rightarrow \hat{\mathcal{E}}_{k,0}$ can be reached. Up to the first order of θ and η , the contribution of the term $\eta/G\hbar$ of \mathcal{J}^* in Eq. (28) is about $\eta/G\hbar \sim 10^{-5}(10^{-2})$. The contribution of the other term $G\theta/4\hbar$ is about $G\theta/4\hbar \sim 10^{-36}(10^{-39})$ which can be neglected. For normal quantum mechanics in commutative space, $\theta = \eta = 0$, in the limit of $H_k \rightarrow \mathcal{E}_{k,0}$, the lowest angular momentum $\mathcal{J}_0^* = \hbar/2$. The corresponding changes of the lowest angular momentum $\Delta\hat{\mathcal{J}}_0^* \equiv \mathcal{J}_0^* - \hat{\mathcal{J}}_0^*$ originated from spatial noncommutativity are, respectively,

$$\Delta \hat{\mathcal{J}}_0^* \sim 10^{-5} \hbar (10^{-2} \hbar). \quad (42)$$

These results are signals of spatial noncommutativity which are within limits of the measurable accuracy of the modified EMS. In this case a positive experimental result shows primary evidence of spatial noncommutativity. One point that should be emphasized is that in this case a negative one cannot draw a conclusive preclusion of spatial noncommutativity, but provides an improved upper bound of η .

The limit of the mechanical kinetic energy approaching its lowest eigenvalue is an important ingredient to obtain the final results. Now we discuss the consequences if this condition is not fulfilled. In this case the effects of spatial noncommutativity are contributed by the second term $\xi^2 \theta p_i p_i / 2\hbar$ and the third term $\xi^2 \eta x_i x_i / 2\hbar$ of Eq. (7). They can be estimated as follows. We consider the ion of a mass number in the order $A \sim 10^2$. Using laser cooling to reduce its average velocity to the order $\bar{v} \sim 10^2 \text{ ms}^{-1}$. Its average momentum is around the order $\bar{p} \sim 10^{-6} \text{ eV/ms}^{-1}$. The average coordinate \bar{x} can be roughly estimated by the uncertainty relation: $\bar{x} \sim \Delta x$, $\bar{p} \sim \Delta p$, and $\bar{x} \sim \hbar / \bar{p}$. Up to the first order of θ and η , the contributions of $\theta \bar{p}^2 / \hbar$ and $\eta \bar{x}^2 / \hbar \sim \eta \hbar / \bar{p}^2$, according to the existing upper bounds of θ and η , are, respectively, about

$$\theta \bar{p}^2 / \hbar \sim 10^{-20} \hbar, \quad \eta \bar{x}^2 / \hbar \sim 10^{-17} \hbar.$$

They are extremely small. Testing contributions of spatial noncommutativity at such a level is almost impossible in the foreseeable future.

Technical difficulties involved in the modified EMS ($I, 2I$) reaction are as follows:

- (i) The differential reaction cross section for the ion-ion scattering is very small. For a rough estimation using a hard sphere model with radius 10^{-10} m , the total cross sections is about 10^{-20} m^2 which depends on the ion's type and energy [25].
- (ii) The efficiency of the coincidence measurements of two out-going ions is quite small. The efficiency of measuring one out-going ion is determined by the geometry of the spectrometer and the open solid angles of ions going out from the trap. As a safe region, the open solid angle is estimated as $\sim 1\%$ of the full 4π without deteriorating the performance of the trap [26]. Therefore, the efficiency of the coincidence measurements of the two out-going ions is about 10^{-4} .

It turns out that, like neutrino experiments, the period of the modified EMS experiment is long. In order to measure one momentum spectrum in a year, one coincidence measurement per day which corresponds to a frequency about 10^{-5} s^{-1} is necessary. For this purpose the number of trapped ions and the incident ion current are, respectively, about 10^{10} and 10^9 s^{-1} . In such cases influence of ion's electric charge and magnetic fields of moving ions are large which need to be greatly reduced by special compensation techniques.

- (iii) Ions with low energy are easily influenced by the disordered background electric and magnetic fields. Thus determinations of momentum distributions of ions through measuring orbits of scattered ions may lead to large error which should be controlled at a reasonable level.
- (iv) In order to guarantee that the inner structures of the target and the incident ions are not changed during scattering a careful selection of the suitable ion's type and beam energy are necessary.

A more detailed analysis by means of the knowledge of the noncommutative wave functions shows that the total cross sections for the ion-ion scattering keeps the same order as one of using a hard sphere model. It is clarified that detailed calculations do not change the basic characteristics of the results of the above qualitative analysis. But a detailed analysis of some experimental technique points are out of the scope of the present theoretical paper.

Similar to EMS, the modified EMS is one of the most subtle processes which provides a variety of information for evaluating the dynamic mechanism of the ($I, 2I$) reaction. Though test of spatial noncommutativity using modified EMS is a challenging enterprise, unlike experiments near the Planck scale 10^{19} GeV , modified EMS provides a realizable way for measuring noncommutative quantum effects on normal quantum scales. It is expected that the experimental realization of the proposal will be the first step towards a conclusive test of spatial noncommutativity.

J. Z. Z's work is supported by NSFC (No. 10575037) and SEDF; L. K. G's work is supported by NSFC (No. 60490280) and NFRPC (No. 2005CB724500); C. G. N's work is supported by NSFC (No. 10575062). J. Z. Z. would like to thank H. P. Liu and X. Shan for helpful discussions.

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